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Sinai-Ruelle-Bowen Measures for Lattice Dynamical Systems

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For weakly coupled expanding maps on the unit circle, Bricmont and Kupiainen showed that the Sinai-Ruelle-Bowen (SRB) measure exists as a Gibbs state. Via thermodynamic formalism, we prove that this SRB measure is indeed the unique equilibrium state for a Hölder continuous potential function on the infinite dimensional phase space. For a more general class of lattice systems that are small perturbations of the uncoupled map lattice, we present the variational principle, the entropy formula, and the formula for the potential function for the SRB measures. For coupled map lattices with nearest neighbor interactions, we give an explicit formula of the potential function for the SRB measure and consequently, obtain the entropy in terms of coupling parameters.

KEY WORDS: SRB measure; lattice dynamical system; thermodynamic formalism; entropy.

1. INTRODUCTION

During the past years, many efforts have been made to extend the concept of the SRB-measure from finite dimensional smooth dynamical systems to spatially extended infinite dimensional dynamical systems.[4, 6–8, 14, 17, 20] In particular, for a general class of weakly coupled expanding maps on the unit circle, Bricmont and Kupiainen showed that the SRB measure exists as a Gibbs state on a phase space of a mixed type: lattice spin systems with both finite spins and infinite spins (a compact metric space). Their proof was general enough to include the case where the coupling is not spatially translation invariant. However, using this approach of construction of the SRB measure, it is difficult to verify that the measure satisfies the

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variational principle and to obtain the (spatiotemporal) entropy of coupled map lattices.

In this paper, we first show that indeed, the SRB measure constructed in ref. 7 satisfies the variational principle by using a more traditional approach: constructing symbolic representations of weakly coupled map lattices using the Markov partition. This approach was used in ref. 20 to establish similar results for weakly coupled hyperbolic systems. The advantage of using this approach here is that it is much simpler since the local map has only an expansive direction even though we have to deal with the non-invertibility of the map. The another advantage is that we can obtain an explicit entropy formula in terms of coupling parameters. Finally, with only minimal changes to the proof presented in ref. 16, this approach will enable us to obtain the smooth dependence of the SRB measure on the system when the coupled map lattice varies and calculate its derivative, i.e., the linear response function.

The lattice systems we consider here are slightly more general than the standard coupled map lattices. Our model is described by a small perturbation of the uncoupled system F, not necessarily in the form of a composition $G \circ F$ with G a diffeomorphism of the phase space. We note that even on the unit circle, a perturbation of an expanding map f can not in general, be expressed in the form $G \circ f$ with G a diffeomorphism of the circle. With potential applications in mind, our last section deals with coupled map lattices only.

The precise description of the lattice system and a summary of main results of the paper are given in Section 2. After a brief introduction of the SRB-measure for an expanding circle map, we show the existence of such measure for the model. The strategy is to extend thermodynamic formalism to lattice dynamical systems and prove desired results for lattice spin systems of equilibrium statistical mechanics. In order to have symbolic representations of lattice dynamical systems, we prove a structural stability theorem for our lattice systems in Section 3. We prove that there is a conjugacy between the uncoupled map lattice and the slightly perturbed one. The conjugacy helps to construct a Markov partition of lattice dynamical systems and obtain the corresponding lattice spin systems. Section 4 contains the extension of thermodynamic formalism to lattice dynamical systems. The results of Sections 3 and 4 are then applied to show that the SRB-measure exists for lattice dynamical systems and is an equilibrium state satisfying the variational principle. The proof consists mainly of the construction of the potential function. In the last section, as an application, we provide further calculation of the potential function in terms of coupling parameters and subsequently, obtain an explicit formula of entropy.

2. PRELIMINARIES

2.1. Lattice Dynamical Systems

Let $\mathbb{Z}^d$ be the d-dimensional integer lattice. We start with the definition of the phase space $\mathscr{M}$.

$$\mathscr{M} = \bigotimes_{i \in \mathbb{Z}^d} S_i^1$$

with $S_i^1 = S^1$, i.e., $\mathscr{M}$ is the direct product of identical copies of the unit circle. A *Lattice Dynamical System* considered in this paper consists of the phase space $\mathscr{M}$ and a map Φ from $\mathscr{M}$ into itself.

In order to study both types of problems: structural stability and invariant measures of lattice dynamical systems, we need to introduce two types of metrics on the phase space $\mathscr{M}$.

Definition of Metrics. We denote by ρ the supremum metric on $\mathscr{M}$. For any $\bar{x} = (x_i)$, $\bar{y} = (y_i) \in \mathscr{M}$,

$$\rho(\bar{x}, \bar{y}) = \sup_{i \in \mathbb{Z}^d} d(x_i, y_i),$$

where d denotes the canonical distance on the unit circle. With the metric ρ, $\mathscr{M}$ is a Banach manifold modelled on the Banach space $l^{\mathbb{Z}^d}$:

$$l^{\mathbb{Z}^d} = \{\bar{x} = (x_i) : \sup_{i \in \mathbb{Z}^d} |x_i| < \infty,\ x_i \in \mathbb{R}\}$$

The Banach space $l^{\mathbb{Z}^d}$ also serves as the universal covering space for $\mathscr{M}$. When we discuss local properties, such as continuity and differentiability, of maps on $\mathscr{M}$, we identify these maps with their lifts in this covering space. The projection function from $l^{\mathbb{Z}^d}$ onto $\mathscr{M}$ is denoted by $\mathscr{P}$ and we have for each $\bar{x} = (x_i) \in l^{\mathbb{Z}^d}$,

$$\mathscr{P}(\bar{x}) = (\exp i2\pi x_i) \in \mathscr{M}.$$

The other metric ρ_q on $\mathscr{M}$ is in fact, a family of metrics that are compatible with the compact topology on $\mathscr{M}$ induced by the direct product structure. It corresponds to the weak* (coordinatewise convergence) topology in the Banach space $l^{\mathbb{Z}^d}$. Given a constant $0 < q < 1$, $\bar{x} = (x_i)$, $\bar{y} = (y_i) \in \mathscr{M}$,

$$\rho_q(\bar{x}, \bar{y}) = \sup_{i \in \mathbb{Z}^d} q^{|i|} d(x_i, y_i),$$

where

$$|i| = |i_1| + |i_2| + \cdots + |i_d|, \qquad i = (i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d.$$

Clearly, the lattice dynamical system $(\mathcal{M}, \Phi)$ just defined is infinite-dimensional. A simple example of Φ is the direct product of identical maps on S^1:

$$F = \bigotimes_{i \in \mathbb{Z}^d} f_i$$

with $f_i = f$ being any differentiable (usually, at least $C^{1+\alpha}$) expanding map on the circle. The degree of the map f is denoted by p, which means that every $x \in S^1$ has precisely p preimages. We always assume that $|f'(x)| > 1$, $x \in S^1$. In this paper, we consider the dynamics of a special class of maps that are small perturbations of such map F.

We describe the class of perturbations with the help of these two kinds of metrics.

Definition of the Perturbation

(C1) *Hölder Continuity Condition in the Metric ρ_q.* We assume that Φ is Hölder continuous with respect to ρ_q for some fixed constant $0 < q < 1$. I.e., there exist $C_1 > 0$ and $\alpha > 0$ such that

$$\rho_q(\Phi(\bar{x}), \Phi(\bar{y})) \leqslant C_1 \rho_q^{\alpha}(\bar{x}, \bar{y}).$$

This Hölder continuity in the metric ρ_q is slightly weaker than the so-called the exponential decay property of the perturbation

$$d(\Phi_i(\bar{x}), \Phi_i(\bar{y})) \leqslant C\theta^{|i-k|} d(x_k, y_k),$$

where Φ_i is the projection of Φ on the lattice site $i \in \mathbb{Z}^d$ and all components of $\bar{x}$ and $\bar{y}$ are the same except at the lattice site k (see ref. 15, Lemma 1). We emphasize that this continuity condition in the metric ρ_q must be imposed before we can describe other conditions on the derivative operator of Φ using its partial derivatives. It is a fact in functional analysis that some bounded linear functionals on the Banach space $l^{\mathbb{Z}^d}$ can fail to be weak* continuous and thus, can not be expressed as an infinite sequence using its values at the weak* basis.[25]

Proposition 1. Assume that $\Phi = (\Phi_i)_{i \in \mathbb{Z}^d}$ is continuous with respect to the metric ρ_q. Assume that Φ is continuously differentiable (C^1) and the sum of partial derivatives $\sum_{j \in \mathbb{Z}^d} |\frac{\partial \Phi_i}{\partial x_j}|_{\bar{x}}| < M$ for some constant M converges uniformly in both $\bar{x}$ and i. Then, the derivative operator $D\Phi$ can be represented by the infinite matrix $(\frac{\partial \Phi_i}{\partial x_j})_{i, j \in \mathbb{Z}^d}$, i.e., for any vector $\bar{y}$ in the tangent space of $\mathscr{M}$ at $\bar{x}$

$$D\Phi_{\bar{x}} \bar{y} = \sum_{j \in \mathbb{Z}^d} \frac{\partial \Phi_i}{\partial x_j}\bigg|_{\bar{x}} y_j.$$

Proof. For any given $\epsilon > 0$, we need to show that

$$\frac{1}{\delta}\left|\Phi_i(\bar{x}+\delta\bar{y}) - \Phi_i(\bar{x}) - \delta \sum_{j \in \mathbb{Z}^d} \frac{\partial \Phi_i}{\partial x_j}\bigg|_{\bar{x}} y_j\right| < \epsilon$$

when $|\delta|$ is sufficiently small. For convenience, we introduce a total order in $\mathbb{Z}^d$: $i < j$ whenever $|i| < |j|$. When $|i| = |j|$, we use the lexicographic order. For example, when $d = 3$, $(0, 0, 1) < (0, 1, 0)$ and $(0, 0, 3) < (1, 1, 1) < (3, 0, 0)$. Let T denote the order preserving one-to-one map from the set of non-negative integers to $\mathbb{Z}^d$. We define a sequence of elements $\bar{z}_k \in \mathscr{M}$, $k = 0, 1, 2$, in the following way: Let $\bar{z}_k(j)$ be the component of $\bar{z}_k$ at lattice site $j \in \mathbb{Z}^d$. Then,

$$\bar{z}_k(j) = \delta y_j, \qquad \text{when} \quad j \leqslant T(k), \quad \bar{z}_k(j) = 0, \qquad \text{when} \quad j > T(k).$$

We have $\lim_{k \to \infty} \bar{x} + \bar{z}_k = \bar{x} + \delta\bar{y}$ in the metric ρ_q. Thus,

$$\Phi_i(\bar{x}+\delta\bar{y}) - \Phi_i(\bar{x}) = \Phi_i(\bar{x}+\bar{z}_0) - \Phi_i(\bar{x}) + \sum_{k=1}^{\infty} [\Phi_i(\bar{x}+\bar{z}_k) - \Phi_i(\bar{x}+\bar{z}_{k-1})].$$

Note that $\bar{x}+\bar{z}_k$ and $\bar{x}+\bar{z}_{k-1}$ differ only at the lattice site $T(k)$. By Mean Value Theorem, for $k = 0, 1, 2,...,$

$$\Phi_i(\bar{x}+\bar{z}_k) - \Phi_i(\bar{x}+\bar{z}_{k-1}) = \frac{\partial \Phi_i}{\partial x_{T(k)}}\bigg|_{\bar{\xi}_k} (\delta y_{T(k)})$$

for some point $\bar{\xi}_k$ between $\bar{x}+\bar{z}_k$ and $\bar{x}+\bar{z}_{k-1}$ $(\bar{x}+\bar{z}_{-1} \equiv \bar{x})$. Thus,

$$\frac{1}{\delta}\left|\Phi_i(\bar{x}+\delta\bar{y}) - \Phi_i(\bar{x}) - \delta \sum_{j \in \mathbb{Z}^d} \frac{\partial \Phi_i}{\partial x_j}\bigg|_{\bar{x}} y_j\right| = \left|\sum_{k=0}^{\infty} \left[\frac{\partial \Phi_i}{\partial x_{T(k)}}\bigg|_{\bar{\xi}_k} - \frac{\partial \Phi_i}{\partial x_{T(k)}}\bigg|_{\bar{x}}\right] y_{T(k)}\right|.$$

Note that for each fixed k, we have

$$\lim_{\delta \to 0} \frac{\partial \Phi_i}{\partial x_{T(k)}}\bigg|_{\bar{\xi}_k} - \frac{\partial \Phi_i}{\partial x_{T(k)}}\bigg|_{\bar{x}} = 0$$

since $\bar{\xi}_k \to \bar{x}$ in the metric ρ as $\delta \to 0$ and the convergence of $\sum_{j \in \mathbb{Z}^d} |\frac{\partial \Phi_i}{\partial x_j}|_{\bar{x}}|$ is uniform in $\bar{x}$. Therefore,

$$\lim_{\delta \to 0} \sum_{k=0}^{\infty} \left[\frac{\partial \Phi_i}{\partial x_{T(k)}}\bigg|_{\bar{\xi}_k} - \frac{\partial \Phi_i}{\partial x_{T(k)}}\bigg|_{\bar{x}} \right] y_{T(k)} = 0. \quad \blacksquare$$

(C2) *Differentiability Condition.* We assume that Φ is at least C^1 with respect to the metric ρ.

(C3) *Small Perturbation Condition.* Φ is C^1-close to F. In terms of partial derivatives, we have

$$\sup_{i \in \mathbb{Z}^d, \bar{x} \in \mathcal{M}} |\Phi_i(\bar{x}) - f(x_i)| + \sup_{i \in \mathbb{Z}^d, \bar{x} \in \mathcal{M}} \sum_{j \in \mathbb{Z}^d} \left| \frac{\partial \Phi_i}{\partial x_j}\bigg|_{\bar{x}} - \frac{\partial F_i}{\partial x_j}\bigg|_{\bar{x}} \right| < \epsilon,$$

for a small constant $0 < \epsilon < 1$.

(C4) *Decaying Coupling Condition.* For $i, j \in \mathbb{Z}^d$, $i \neq j$,

$$\left| \frac{\partial \Phi_i}{\partial x_j} \right| < C_3 \epsilon e^{-\beta |i-j|},$$

where $C_3 > 0$ and $\beta > 0$ are constants.

Remark 1

(1) Conditions (C1), (C2), and (C4) are sufficient for Proposition 1 to hold.

(2) One can formulate different types of decay conditions other than using ϵ and β in (C4) for the partial derivatives. However, it seems that the exponential decay of the coupling between remote lattice sites is necessary for the study of SRB measures via thermodynamic formalism in later sections. This exponential decay condition also allows a simple proof of the Hölder continuity of the conjugating map in the metric ρ_q in the next section. This type of assumptions appeared in previous papers such as refs. 4, 6, 7, 12, 15, 16, 20, and 28. Other types of decay of coupling are also possible when one uses the transfer operator approach (see refs. 14 and 30). But it is unclear if the SRB measure in refs. 14 and 30 will satisfy the variational principle. The condition (C4) can also be formulated in terms of the Lipschitz continuity in weighted metrics.[15, 20]

The last condition (C5) concerns the smallness of the derivative of the perturbation. This condition will not be needed until the proof of the uniqueness of SRB measures as an equilibrium state. For $\bar{x} = \in \mathscr{M}$, $i, j \in \mathbb{Z}^d$, let

$$a_{ii}(\bar{x}) = \frac{\frac{\partial \Phi_i}{\partial x_i}\big|_{\bar{x}}}{f'(x_i)} - 1 \qquad \text{and} \qquad a_{ij}(\bar{x}) = \frac{\partial \Phi_i}{\partial x_j}\bigg|_{\bar{x}}, \quad i \neq j.$$

(C5) ***Hölder Condition on the Derivative.*** For all $i, j, k \in \mathbb{Z}^d$ and $\bar{x} = (x_l)$, $\bar{y} = (y_l) \in \mathscr{M}$ with $x_l = y_l$, $l \in \mathbb{Z}^d$, $l \neq k$,

$$|a_{ij}(\bar{x}) - a_{ij}(\bar{y})| < C_4 \epsilon e^{-\beta |i-k|} d^{\alpha}(x_k, y_k),$$

for some constant C_4 and $0 < \alpha < 1$.

This condition is a little weaker than a similar condition used in ref. 7 (expression (4)) for coupled map lattices and is the same as the one stated in ref. 20 (expression (7)).

Other Definitions

(1) ***Spatial Translation Invariance.*** Let σ_s^k, $k \in \mathbb{Z}^d$ denote the map induced by shifts (or translations) on the lattice $\mathbb{Z}^d$: $(\sigma_s^k(\bar{x}))_i = x_{i+k}$. When $d = 1$, this is just the leftward shift. Φ is called shift (or translation) invariant if Φ and σ_s^k commute: $\Phi \circ \sigma_s^k = \sigma_s^k \circ \Phi$.

(2) ***Finite Volume Approximation of Φ.*** For each finite volume $V \subset \mathbb{Z}^d$, $M_V = \bigotimes_{i \in V} S_i^1$. Fix a point $\bar{x}^* = (x_i^*)_{i \in \mathbb{Z}^d} \in \mathscr{M}$. For convenience, we shall take $\bar{x}^* = (0)_{i \in \mathbb{Z}^d} \in \mathscr{M}$, i.e., the origin. The map Φ_V denotes the following map from M_V to itself:

$$(\Phi_V(x_V))_i = (\Phi(x_V, x_{\hat{V}}^*))_i, \quad i \in V,$$

where $\hat{V} = \mathbb{Z}^d \backslash V$, the complement of V in $\mathbb{Z}^d$. The structural stability theorem[31] tells us that when the perturbation is sufficiently small, Φ_V is an expanding map on M_V conjugated by a continuous map h_V:

$$\Phi_V \circ h_V = h_V \circ F_V.$$

With a little abuse of notation, we also use Φ_V to denote the following extended map on $\mathscr{M}$:

$$(\Phi_V(\bar{x}))_i = \begin{cases} (\Phi(x_V, x_{\hat{V}}^*))_i, & i \in V \\ f(x_i), & i \in \hat{V} \end{cases} \tag{2.1}$$

i.e., the perturbation is restricted inside the finite volume V. It is easy to see that whenever Φ satisfies conditions (C1)–(C5), Φ_V satisfies the same conditions with the same set of constants.

2.2. The Sinai-Ruelle-Bowen Measure

For expanding maps on the circle S^1, The Sinai-Ruelle-Bowen measure has many equivalent descriptions. We list two of them whose extensions to the lattice dynamical systems are discussed in this article.

(1) *Weak* Limit of Iterates of Lebesgue Measure.* Let $A \subset S^1$ be any Borel set, f be a $C^r, r > 1$ expanding map, and λ be the normalized Lebesgue measure (or any probability measure equivalent to λ) on the circle. Then, the following limit exists:

$$\mu_f(A) = \lim_{n \to \infty} \lambda(f^{-n}(A)).$$

The limiting measure μ_f is invariant under f: $\mu_f(A) = \mu_f(f^{-1}(A))$.

(2) *Variational Principle.* Let Γ be the set of all invariant probability measures on S^1 with respect to f and $h_\gamma(f)$ be the measure theoretical entropy w.r.t $\gamma \in \Gamma$. Then,

$$\sup_{\gamma \in \Gamma} \left(h_\gamma(f) + \int_{S^1} -\log |f'(x)| \, d\gamma \right) = 0.$$

There exists a unique measure μ_f at which the supremum is attained. Any probability measure satisfies the equality is called an equilibrium state for the potential function $-\log |f'(x)|$. Definition of an equilibrium measure for arbitrary continuous function can be found in ref. 29.

The measures obtained from these two procedures are the same and are called the Sinai-Ruelle-Bowen (SRB) measure for the expanding map f. The SRB measure can also be defined as the fixed point of the Perron-Frobenius operator. Another way to define the measure is to construct a Markov partition for the expanding map and obtain the measure as a Gibbs state through a sequence of conditional probabilities.[33]

Measures with similar properties exist for other maps, e.g., transitive Anosov maps. For general C^r-expanding ($r > 1$) maps on closed manifolds, the description is almost identical. For instance, in our context, Φ_V is an expanding map on M_V. Thus, there exists an SRB-measure μ_V on M_V which is invariant under Φ_V, absolutely continuous with respect to the Lebesgue measure, and is mixing. This SRB-measure is the unique invariant measure satisfies the Variational principle:

$$h_{\mu_V}(\Phi_V) = \int \log J\Phi_V(x_V) \, d\mu_V, \tag{2.2}$$

where $h_{\mu_V}(\Phi_V)$ is the entropy of Φ_V with respect to μ_V and $J\Phi_V(x_V)$ is the Jacobian of Φ_V.

For coupled map lattices, it has been shown in ref. 7 using the transfer operator method with the cluster expansion technique that the (thermodynamic) limit of the measure μ_V exists as the volume V goes to $\mathbb{Z}^d$. The limiting measure μ is a Gibbs state invariant under Φ and it is exponentially mixing with respect to both Φ and σ_s.

In this article, we will show that the measure μ also satisfies the variational principle. The main results of this paper are the following.

Theorem 1. Assume that the map Φ satisfies conditions (C1)–(C5) for sufficiently small ϵ and is translation invariant.

(1) The thermodynamic limit of SRB-measures μ_V exists. The limiting measure μ is invariant and exponentially mixing with respect to both Φ and spatial translations: For any Hölder continuous functions ϕ and ψ in the metric ρ_q on $\mathcal{M}$,

$$\lim_{n+|k|\to\infty} \int_{\mathcal{M}} \phi(\Phi^n \sigma_s^k \bar{x})\, \psi(x)\, d\mu - \int_{\mathcal{M}} \phi(\bar{x})\, d\mu \int_{\mathcal{M}} \psi(x)\, d\mu = 0. \tag{2.3}$$

(2) The measure μ is the unique equilibrium state for a Hölder continuous (in the metric ρ_q) potential function $\varphi(\bar{x})$ close to $-\log|f'|$ under the $\mathbb{Z}_+^{d+1}$-action τ generated by Φ and spatial translations. Moreover, the entropy formula holds:

$$P_\tau(\varphi) = h_\mu(\tau) + \int \varphi\, d\mu = 0.$$

(3) The (spatiotemporal) entropy of τ with respect to μ is the limit of the average entropy of Φ_V over the volume V:

$$h_\mu(\tau) = \lim_{V\to\mathbb{Z}^d} \frac{1}{|V|} h_{\mu_V}(\Phi_V).$$

(4) The potential function $\varphi(\bar{x})$ of the SRB measure μ with respect to the action τ is given by

$$\varphi(\bar{x}) = -\log|f'(x_0)| + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} a_{00}^{(n)}(\bar{x}), \tag{2.4}$$

where $a_{00}^{(n)}(\bar{x})$ is the entry of the infinite matrix A^n corresponding to the (0, 0) lattice point of $\mathbb{Z}^d \times \mathbb{Z}^d$ and the matrix $A(\bar{x})$ is defined by the relation

$$\left(\frac{\partial \Phi_i}{\partial x_j}(\bar{x})\right)_{i,j\in\mathbb{Z}^d} = (\mathrm{diag}(f'(x_i)))(I + A(\bar{x})).$$

We outline the steps of the proofs.

Step 1. Prove that the map Φ and F are conjugate by a continuous map h. Show this map h has special regularities: it is Hölder continuous in the metric ρ_q and the conjugating map h_V between Φ_V and F_V converges to h uniformly in the metric ρ_q.

Step 2. Pull back the SRB measures μ_V for the finite dimensional systems (Φ_V, M_V) onto the symbolic representations induced by the Markov Partition to obtain equilibrium states ν_V.

Step 3. Show that the equilibrium states as Gibbs states ν_V converge to an equilibrium state ν on a $(d+1)$-dimensional lattice spin system. The potential function for ν is obtained by localizing the potential functions of ν_V. The uniqueness and the exponential mixing property follow from special Hölder continuity of this potential function in the metric ρ_q.

Step 4. Push forward the measure ν onto $\mathscr{M}$ and show that this measure is the unique equilibrium state for a corresponding potential function.

Remark 2

(1) The proof of the entropy formula is the same as that for the coupled hyperbolic attractors and therefore, is omitted.

(2) The construction of the potential function (2.4) first appeared in ref. 7 (expression (14) on p. 719) for coupled expanding map lattices.

(3) When the local map f is C^r for $r > 4$ and Φ is restricted to a C^r-neighborhood of F with its partial derivatives up to order 4 satisfying decay conditions similar to Condition (C4), one can prove that the conjugating map h depends smoothly on the perturbation Φ. Consequently, the potential function $\varphi(h(\bar{x}))$ in (2.4) and the SRB measure μ depend on Φ smoothly. Proofs are given for coupled hyperbolic systems in ref. 16 and are essentially identical for our systems here (only simpler) and thus, are omitted.

3. STRUCTURAL STABILITY AND REGULARITY OF THE CONJUGATING MAP

In this section, we prove structural stability for the map F and prove the regularity of the conjugating map that will play an important role in studying invariant measures for such systems. The proof of structural stability follows closely the one in the finite dimensional case (ref. 31). The Hölder regularity of the conjugating map is proved via finite dimensional approximation.

Let $\mathscr{P}$ be the projection from the covering space $l^{\mathbb{Z}^d}$ of $\mathscr{M}$ onto $\mathscr{M} = \bigotimes_{n \in \mathbb{Z}^d} S^1$. Let $\tilde{F}$ and $\tilde{\Phi}$ denote lifts of F and Φ in the covering space $l^{\mathbb{Z}^d}$, respectively, i.e., both $\tilde{F}$ and $\tilde{\Phi}$ are continuous and satisfy

$$\mathscr{P} \circ \tilde{F} = F \circ \mathscr{P}; \qquad \mathscr{P} \circ \tilde{\Phi} = \Phi \circ \mathscr{P}.$$

Note that lifts of F and Φ are not unique. To fix lifts for F and Φ, we assume that $\tilde{F}(0) = 0$, and $\rho(\tilde{\Phi}(0), 0) \leqslant \epsilon < \frac{1}{2}$. Under the supremum distances, a map on $\mathscr{M}$ and its lift in $l^{\mathbb{Z}^d}$ are locally identical. For $\bar{x}$ and $\bar{y}$ close in $l^{\mathbb{Z}^d}$, $\rho(\tilde{\Phi}(\bar{x}), \tilde{\Phi}(\bar{y})) = \rho(\Phi(\mathscr{P}\bar{x}), \Phi(\mathscr{P}\bar{y}))$. Therefore, if Φ and F satisfy conditions (C1)–(C5), $\tilde{\Phi}$ and $\tilde{F}$ satisfy the same conditions with only a small modification to (C1): (C1) holds for $\tilde{\Phi}$ when $\rho(\bar{x}, \bar{y}) < 1$ in $l^{\mathbb{Z}^d}$. For simplicity, we shall use same notations for corresponding objects on $\mathscr{M}$ and its covering space $l^{\mathbb{Z}^d}$.

Under the conditions (C1)–(C3), the conjugacy between the lifted maps $\tilde{\Phi}$ and $\tilde{F}$ can be proved by using the fixed point theorem for contracting maps. The conjugacy between Φ and F follows immediately.

Theorem 2 (Structural Stability). Assume that the map Φ satisfies conditions (C1)–(C3) for a sufficiently small ϵ. Then, Φ is topologically conjugate to F: there exists a homeomorphism $h: \mathscr{M} \to \mathscr{M}$ such that

$$\Phi \circ h = h \circ F.$$

Proof. We first observe that $\tilde{F}$ and $\tilde{\Phi}$ satisfy the following translation conditions.

$$\tilde{F}(\bar{x} + \bar{n}) = \tilde{F}(\bar{x}) + p\bar{n}, \qquad \tilde{\Phi}(\bar{x} + \bar{n}) = \tilde{\Phi}(\bar{x}) + p\bar{n}, \tag{3.1}$$

where $\bar{x} \in l^{\mathbb{Z}^d}$, p is the degree of the map f, and $\bar{n} \in \mathbb{Z}^d$. The first equation is obvious. To see that the second equation holds, let $\bar{e}_0 = (n_i)_{i \in \mathbb{Z}^d}$ with $n_i = 0$ for all $i \neq 0$ and $n_0 = 1$. We consider the straight line connecting two points

$\bar{x}$ and $\bar{x}+\bar{e}_0$ in $l^{\mathbb{Z}^d}$. Note that the projection of this line onto $\mathcal{M}=\bigotimes_{i\in\mathbb{Z}^d} S^1$ is a circle since $\mathcal{P}\bar{x}=\mathcal{P}(\bar{x}+\bar{e}_0)$. Since $\mathcal{P}\circ\tilde{\Phi}=\Phi\circ\mathcal{P}$, we have

$$\mathcal{P}\tilde{\Phi}(\bar{x}+\bar{e}_0)=\mathcal{P}(\tilde{\Phi}(\bar{x})),$$

which means

$$\tilde{\Phi}(\bar{x}+\bar{e}_0)=\tilde{\Phi}(\bar{x})+(m_i)_{i\in\mathbb{Z}^d}$$

for some integer sequence $(m_i)_{i\in\mathbb{Z}^d}$, $m_i\in\mathbb{Z}$. Since $\tilde{\Phi}(\bar{x}+\bar{e}_0)$ is close to $\tilde{F}(\bar{x}+\bar{e}_0)=\tilde{F}(\bar{x})+p\bar{e}_0$, we must have

$$\tilde{\Phi}(\bar{x}+\bar{e}_0)=\tilde{\Phi}(\bar{x})+p\bar{e}_0.$$

Thus, the second equation in (3.1) follows from the coordinate-wise continuity (w^*-continuity) of $\tilde{\Phi}$.

By equations in (3.1), it is easy to verify that both $\tilde{F}$ and $\tilde{\Phi}$ are invertible, differentiable, and expanding maps. Now we consider the complete metric space $C_P(l^{\mathbb{Z}^d}, l^{\mathbb{Z}^d})$ (the subscript P indicates certain periodicity) consisting of all continuous maps $g(\bar{x})$ from $l^{\mathbb{Z}^d}$ to itself satisfying the condition $g(\bar{x}+\bar{n})=g(\bar{x})+\bar{n}$:

$$C_P(l^{\mathbb{Z}^d}, l^{\mathbb{Z}^d})=\{g\colon l^{\mathbb{Z}^d}\to l^{\mathbb{Z}^d}, \text{continuous}, g(\bar{x}+\bar{n})=g(\bar{x})+\bar{n}, \text{for } \bar{n}\in\mathbb{Z}^d\}. \tag{3.2}$$

The metric on this space is the supremum metric induced by the metric ρ on $l^{\mathbb{Z}^d}$:

$$\rho(g_1, g_2)=\sup_{\bar{x}\in l^{\mathbb{Z}^d}} \rho(g_1(\bar{x}), g_2(\bar{x})).$$

Define a map $\mathcal{L}_\Phi$ on $C_P(l^{\mathbb{Z}^d}, l^{\mathbb{Z}^d})$ by

$$\mathcal{L}_\Phi g(\bar{x})=\tilde{\Phi}^{-1}\circ g\circ\tilde{F}(\bar{x}). \tag{3.3}$$

To see that the map $\mathcal{L}_\Phi$ is well-defined we need to observe that $\mathcal{L}_\Phi g(\bar{x})$ is continuous. The relation

$$\mathcal{L}_\Phi g(\bar{x}+\bar{n})=\mathcal{L}_\Phi g(\bar{x})+\bar{n}$$

follows directly from equations in (3.1).

To obtain the conjugating map, we need to show that the map $\mathscr{L}_\Phi$ has a fixed point near the identity map of the space $l^{\mathbb{Z}^d}$: $\mathrm{Id} \in C_P(l^{\mathbb{Z}^d}, l^{\mathbb{Z}^d})$. For any map $g \in C_P(l^{\mathbb{Z}^d}, l^{\mathbb{Z}^d})$ with $\rho(g, \mathrm{Id}) < \delta$ $(0 < \delta < 1)$.

$$\begin{aligned}\rho(\mathscr{L}g(\bar{x}), \mathrm{Id}(\bar{x})) &= \rho(\tilde{\Phi}^{-1} \circ g \circ \tilde{F}(\bar{x}), \tilde{\Phi}^{-1} \circ \mathrm{Id} \circ \tilde{\Phi}(\bar{x})) \\ &\leqslant l\rho(g, \tilde{\Phi} \circ \tilde{F}^{-1}) \leqslant l[\rho(g, \mathrm{Id}) + \rho(\mathrm{Id}, \tilde{\Phi} \circ \tilde{F}^{-1})] \leqslant l(\delta + \epsilon),\end{aligned}$$

where $0 < l < 1$ denotes the Lipschitz constant for $\tilde{\Phi}^{-1}$. For any fixed $0 < \delta < 1$, we can choose ϵ in (C3) sufficiently small such that $l(\delta + \epsilon) < \delta$. i.e., the map $\mathscr{L}_\Phi$ maps the δ-neighborhood of the identity map Id in $C_P(l^{\mathbb{Z}^d}, l^{\mathbb{Z}^d})$ into itself. Since $l < 1$, $\mathscr{L}_\Phi$ is also contracting. Therefore, there exists a unique fixed point $\tilde{h}$ in this δ-neighborhood of Id. The map $\tilde{h}$ satisfies the equation $\tilde{\Phi} \circ \tilde{h} = \tilde{h} \circ \tilde{F}$. In fact, we have

$$\tilde{h} = \lim_{n \to \infty} \mathscr{L}_\Phi^n(\mathrm{Id}). \tag{3.4}$$

To show that $\tilde{h}$ is a homeomorphism of the Banach space $l^{\mathbb{Z}^d}$, we need to apply the same argument to the map

$$g \to \tilde{F}^{-1} \circ g \circ \tilde{\Phi}$$

to obtain its fixed point $\tilde{h}'$ close to the identity. We note that $\tilde{h}' \circ \tilde{h}$ is close to the identity and is the unique fixed point of the map

$$g \to \tilde{F}^{-1} \circ g \circ \tilde{F}.$$

Thus, it must be the identity map, i.e.,

$$\tilde{h}' \circ \tilde{h} = \mathrm{Id}.$$

Similarly, we have

$$\tilde{h} \circ \tilde{h}' = \mathrm{Id}.$$

Thus, $\tilde{h}$ conjugates $\tilde{\Phi}$ and $\tilde{F}$.

The projection of $\tilde{h}$ onto $\mathscr{M}$ is the conjugating map h between F and Φ. ∎

Study of the metric properties (existence, uniqueness of invariant measures) of lattice dynamical systems requires additional properties of the conjugating map h. Since the phase space $\mathscr{M}$ is not compact under the supremum metric ρ, it is not convenient for us to study invariant measures. The natural topology under which invariant measures can be rather easily studied is the product topology on $\mathscr{M}$. The exponential decay condition

(C4) guarantees that the conjugating map h will have the desired regularity in the product topology to transport invariant measures of the unperturbed system $(\mathcal{M}, F)$ onto the perturbed system $(\mathcal{M}, \Phi)$.

Theorem 3

(1) If Φ satisfies conditions (C1), (C2), and (C4), then Φ is Lipschitz continuous with respect to the metric ρ_q for any q with $e^{-\beta} < q < 1$.

(2) When $\epsilon > 0$ in (C3)–(C4) is sufficiently small, $\tilde{\Phi}^{-1}$ is also Lipschitz continuous and contracting in the metric ρ_q for any q with $e^{-\beta} < q < 1$.

Proof. (1) Let $i \in \mathbb{Z}^d$ be fixed. Then,

$$\begin{aligned} q^{|i|} d(\Phi_i(\bar{x}), \Phi_i(\bar{y})) &\leqslant \sum_{j \in \mathbb{Z}^d} q^{|i|} \left\| \frac{\partial \Phi_i}{\partial x_j} \right\| d(x_j, y_j) \\ &\leqslant q^{|i|} \left\| \frac{\partial \Phi_i}{\partial x_i} \right\| d(x_i, y_i) + \sum_{j \in \mathbb{Z}^d, j \neq i} q^{|i|} \epsilon C_3 e^{-\beta |i-j|} d(x_j, y_j) \\ &\leqslant \left(\sum_{j \neq i} \epsilon C_3 (q e^{\beta})^{-|i-j|} + C \right) \sup_{j \in \mathbb{Z}^d} q^{|j|} d(x_j, y_j), \end{aligned}$$

where C is a constant.

(2) To prove the second part of the theorem, we need only to use the following lemma whose slightly different versions appeared in refs. 15 and 26.

Lemma 1. Let $\tilde{\Phi}_i^{-1}$ denote the coordinate of $\tilde{\Phi}^{-1}$ at $i \in \mathbb{Z}^d$, then for any $0 < \beta' < \beta$, there exists constant $C(\beta')$ such that

$$\left| \frac{\partial \tilde{\Phi}_i^{-1}}{\partial x_j} \right| \leqslant C(\beta')\, \epsilon e^{-\beta' |i-j|}$$

for $i \neq j$, $i, j \in \mathbb{Z}^d$ and

$$\left| \frac{\partial \tilde{\Phi}_i^{-1}}{\partial x_i} \right| \leqslant l < 1,$$

where l is the Lipschitz constant of $\tilde{\Phi}^{-1}$ in the metric ρ.

We repeat the estimation in the proof of (1) for $\tilde{\Phi}^{-1}$ to obtain the desired result. Note that the Lipschitz constant of $\tilde{\Phi}^{-1}$ in the ρ_q metric is less then one when $\epsilon > 0$ in (C3) and (C4) is sufficiently small. We will also denote this Lipschitz constant in the metric ρ_q by l. ∎

The Hölder continuity of Φ and $\tilde{\Phi}^{-1}$ in the metric ρ_q can be passed onto the conjugating map h. The next theorem implies that h is indeed, Hölder continuous in the metric ρ_q. Moreover, it is close to a direct product of maps on the unit circle.

Theorem 4. The conjugating map h satisfies the following properties.

(1) There exist constants $0 < \delta < 1$ and $C > 0$ such that

$$d(h_i(\bar{x}), h_i(\bar{y})) \leqslant C d^{\delta}(x_i, y_i) \tag{3.5}$$

for any $\bar{x} = (x_k)$, $\bar{y} = (y_k) \in \mathcal{M}$ with $x_k = y_k$ for all $k \in \mathbb{Z}^d$ except $k = i$.

(2) For any $0 < \beta' < \beta$, there exists constant $c(\epsilon)$, depending only on ϵ and $c(\epsilon) \to 0$ as $\epsilon \to 0$, such that for any fixed $i, j \in \mathbb{Z}^d$, $i \neq j$ and any $\bar{x}, \bar{y} \in \mathcal{M}$ with $x_k = y_k$ for all $k \in \mathbb{Z}^d$ except $k = j$,

$$d(h_i(\bar{x}), h_i(\bar{y})) \leqslant c(\epsilon)\, e^{-\frac{\beta'}{2}|i-j|} d^{\delta}(x_j, y_j), \tag{3.6}$$

The proof of the first property relies on the finite dimensional approximation while the second part of the theorem, inequality (3.6) is proved by induction. For simplicity, we choose $V = \{i \in \mathbb{Z}^d, |i| \leqslant n\}$. The following lemma can be directly verified using the definitions.

Lemma 2. For any volumes $V \subseteq V' \subseteq \mathbb{Z}^d$,

(1) $$\Phi_V(x_V) = \Phi_{V'}(x_V, x^*_{V' \setminus V}).$$

(2) $$\rho_q(\Phi_V, \Phi_{V'}) \leqslant C q^n$$

In particular, $\rho_q(\Phi_V, \Phi) \leqslant C q^n$, where C is a constant.

Proof of (1) of Theorem 4. First, we observe that the Lipschitz continuity in the metric ρ_q holds for the lifted map $\tilde{\Phi}$ provided that it is considered in a bounded set in the metric ρ.

Next, we consider the extended maps of Φ_V defined on the entire $\mathcal{M}$ using the formula (2.1). Similarly, we extend the lifted maps and we will use the same notations for these extended maps. Since the map Φ_V satisfies conditions (C1)–(C4) with the same set of constants, Theorem 3 holds when Φ is replaced by Φ_V.

For $n = 1, 2, \ldots$, we have a sequence of maps $\mathcal{L}_{\Phi_V}$ defined on the same metric space $C_P(l^{\mathbb{Z}^d}, l^{\mathbb{Z}^d})$:

$$\mathcal{L}_{\Phi_V} g = \tilde{\Phi}_V^{-1} \circ g \circ F.$$

Estimating the induced distance between $\mathscr{L}_{\Phi_V}$ and $\mathscr{L}_{\Phi}$, we have

$$\begin{aligned}\rho_q(\mathscr{L}_{\Phi_V} g, \mathscr{L}_{\Phi} g) &= \rho_q(\tilde{\Phi}_V^{-1} g\tilde{F}, \tilde{\Phi}^{-1} g\tilde{F}) \\ &= \rho_q(\tilde{\Phi}_V^{-1} g\tilde{F}, \tilde{\Phi}_V^{-1}\tilde{\Phi}_V\tilde{\Phi}^{-1} g\tilde{F}) \\ &\leqslant l\rho_q(\tilde{\Phi}\tilde{\Phi}^{-1} g\tilde{F}, \tilde{\Phi}_V\tilde{\Phi}^{-1} g\tilde{F}) \leqslant lCq^n.\end{aligned}$$

Note that all maps $\mathscr{L}_{\Phi_V}$ have the same contracting coefficient l. By the uniform contracting map theorem, the unique fixed points h_V satisfy the inequality

$$\rho_q(h_V, h) \leqslant C'q^n,$$

where the constant C' depends only on l and C. We conclude that h_V converges to h exponentially fast in the metric ρ_q.

To prove the first statement in Theorem 4, we just need to show the inequality holds when $i = 0$ because of the translation invariance. This is achieved by applying the convergence of h_V to h to the following lemma.

Lemma 3. There exist constants $C > 0, \delta > 0$ independent of the volume V, such that

$$\rho_q(h_V(x_V), h_V(y_V)) \leqslant C\rho_q^{\delta}(x_V, y_V),$$

for all $x_V = (x_k)$, $y_V = (y_k) \in M_V$ with $x_k = y_k$, $k \in V$, $k \neq 0$.

Proof of the Lemma. We follow the proof of a similar result for hyperbolic systems on p. 599 in ref. 22. We just need to make sure that the constants involved are independent of the volume V.

Since $\tilde{h}$ is the lift of h and $\mathscr{M}$ is compact in the metric ρ_q, both maps $\tilde{h}$ and h are uniformly continuous with respect to the metric ρ_q. Because the convergence $h_V \to h$ is uniform, we have that for any given $\epsilon_0 > 0$, there exists $\delta_0 > 0$ independent of the volume V such that $\rho_q(h_V(x_V), h_V(y_V)) < \epsilon_0$ whenever $\rho_q(x_V, y_V) < \delta_0$. The same is true for the lifted map $\tilde{h}_V$ in a bounded (in the metric ρ) set.

Let $l < 1$ and $L > 1$ be the Lipschitz constants for $\tilde{\Phi}_V^{-1}$ and $\tilde{F}_V$ (or, f) in the metric ρ_q. Both constants are independent of V. Choose $0 < \delta < 1$ such that $lL^{\delta} < 1$. Let $x_V, y_V \in M_V$ with $x_k = y_k$, $k \in V$, $k \neq 0$. Note that $\rho_q(x_V, y_V) = d(x_0, y_0)$. We may assume that $\rho_q(x_V, y_V) < \delta_0$. Let $m \geqslant 0$ be an integer such that

$$L^m \rho_q(x_V, y_V) < \delta_0 \leqslant L^{m+1}\rho_q(x_V, y_V).$$

Since $\rho_q(\tilde{F}_V^m(x_V), \tilde{F}_V^m(y_V))) \leqslant L^m \rho_q(x_V, y_V) < \delta_0$, we have

$$\begin{aligned}\rho_q(h_V(x_V), h_V(y_V)) &= \rho_q(\tilde{h}_V(x_V), \tilde{h}_V(y_V))\\ &= \rho_q(\tilde{\Phi}_V^{-m}\tilde{h}_V\tilde{F}_V^m(x_V), \tilde{\Phi}_V^{-m}\tilde{h}_V\tilde{F}_V^m(y_V))\\ &\leqslant l^m\epsilon_0 = \frac{\epsilon_0}{\delta_0^\delta} l^m\delta_0^\delta \leqslant \frac{\epsilon_0}{\delta_0^\delta} l^m L^{(m+1)\delta}\rho_q^\delta(x_V, y_V)\\ &\leqslant \frac{\epsilon_0}{\delta_0^\delta} L^\delta \rho_q^\delta(x_V, y_V). \quad \blacksquare\end{aligned}$$

We continue to prove the second part of the theorem.

Proof of the Smallness in the Hölder Coefficients. In the proof of structural stability, we have shown that $\tilde{h}$, the lift of h can be obtained as a limit: $\tilde{h} = \lim_{n\to\infty} \mathscr{L}_\Phi^n(\mathrm{Id})$ in the metric induced by ρ. We now use induction on n. Obviously, the identity Id satisfies the estimation (3.6). Let us assume that $g(\bar{x}) = \mathscr{L}_\Phi^n(\mathrm{Id})$ satisfies these inequalities, i.e.,

$$d(g_i(\bar{x}), g_i(\bar{y})) \leqslant c(\epsilon)\, e^{-\frac{\beta'}{2}|i-j|}\, d^\delta(x_j, y_j).$$

We show that $\mathscr{L}_\Phi g(\bar{x}) = \tilde{\Phi}^{-1} \circ g \circ \tilde{F}(\bar{x})$ satisfies the same estimations when ϵ is sufficiently small. We need estimations of the entries of the derivative matrix $D\tilde{\Phi}^{-1}$ from Lemma 1:

$$\left|\frac{\partial\tilde{\Phi}_i^{-1}}{\partial x_i}\right| \leqslant l < 1 \qquad \text{and} \qquad \left|\frac{\partial\tilde{\Phi}_i^{-1}}{\partial x_k}\right| \leqslant C(\beta')\,\epsilon e^{-\beta'|i-k|}$$

for some constant $\beta' < \beta$. Let L denote the Lipschitz constant of map f. Notice that $\bar{x}$ and $\bar{y}$ differ only at the lattice site j. We have

$$\begin{aligned}&d(\tilde{\Phi}_i^{-1} \circ g \circ \tilde{F}(\bar{x}), \tilde{\Phi}_i^{-1} \circ g \circ \tilde{F}(\bar{y}))\\ &\leqslant \sum_{k\in\mathbb{Z}^d} \left\|\frac{\partial\tilde{\Phi}_i^{-1}}{\partial x_k}\right\| d(g_k \circ \tilde{F}(\bar{x}), g_k \circ \tilde{F}(\bar{y}))\\ &= \left\|\frac{\partial\tilde{\Phi}_i^{-1}}{\partial x_j}\right\| d(g_j \circ \tilde{F}(\bar{x}), g_j \circ \tilde{F}(\bar{y})) + \left\|\frac{\partial\tilde{\Phi}_i^{-1}}{\partial x_i}\right\| d(g_i \circ \tilde{F}(\bar{x}), g_i \circ \tilde{F}(\bar{y}))\\ &\quad + \sum_{k\neq i,j} \left\|\frac{\partial\tilde{\Phi}_i^{-1}}{\partial x_k}\right\| d(g_k \circ \tilde{F}(\bar{x}), g_k \circ \tilde{F}(\bar{y}))\\ &\leqslant \left(C(\beta')\,\epsilon C e^{-\beta'|i-j|} + \left(l + \sum_{k\neq i,j} C(\beta')\,\epsilon e^{-\beta'|i-k|}\right) c(\epsilon)\, e^{-\frac{\beta'}{2}|i-j|}\right) d^\delta(f(x_j), f(y_j))\\ &\leqslant \left(C(\beta')\,\epsilon C e^{-\beta'|i-j|} L^\delta + \left(lL^\delta + \sum_{k\neq i,j} C(\beta')\,\epsilon e^{-\beta'|i-k|} L^\delta\right) c(\epsilon)\, e^{-\frac{\beta'}{2}|i-j|}\right) d^\delta(x_j, y_j).\end{aligned}$$

We first choose δ such that $lL^{\delta}<1$ (this is exactly how δ is chosen in Lemma 3 in the first place). We then need $\epsilon>0$ to be sufficiently small so that

$$C(\beta')\,\epsilon C e^{-\frac{\beta'}{2}|i-j|}L^{\delta}+lc(\epsilon)\,L^{\delta}+\sum_{k\neq i,\,j} C(\beta')\,\epsilon e^{-\beta'\,|i-k|}c(\epsilon)\,L^{\delta}\leqslant c(\epsilon).$$

In fact, we can simply let $c(\epsilon)=\sqrt{\epsilon}$. ▮

As a consequence of Theorem 4, we have the Hölder continuity of h.

Corollary 1. There exist constants $0<\delta<1$ and $C>0$ such that

$$\rho_q(h(\bar{x}),\,h(\bar{y}))\leqslant C\rho_q^{\delta}(\bar{x},\,\bar{y})$$

for all $\bar{x},\,\bar{y}\in\mathscr{M}$.

4. THERMODYNAMIC FORMALISM

In this section, we discuss how to obtain equilibrium states for suitable potentials on the lattice dynamical system $(\Phi,\,\mathscr{M})$ with respect to the $\mathbb{Z}_+^{d+1}$ action using the Gibbs states on its symbolic representation (a lattice spin system). The exposition is similar to that in refs. 15 and 20. So some details are omitted.

4.1. Markov Partition and Semi-Conjugacy

The symbolic representation of the lattice system $(\Phi,\,\mathscr{M})$ is induced by the symbolic representation of the local map f through the conjugating map h.

Since f is an expanding map with degree p, we have a Markov partition and a semi-conjugacy π between f and the left shift σ_t on Σ_p, the (one direction) full shift of p symbols:

$$f\circ\pi=\pi\circ\sigma_t.$$

This semi-conjugacy is extended to a semi-conjugacy $\bar{\pi}=\bigotimes_{i\in\mathbb{Z}^d}\pi$ between F and $\bigotimes_{i\in\mathbb{Z}^d}(\sigma_t)_i$ on $\Sigma_p^{\mathbb{Z}^d}=\bigotimes_{i\in\mathbb{Z}^d}(\Sigma_p)_i$, where $(\Sigma_p)_i$ are copies of Σ_p. This shift $\bigotimes_{i\in\mathbb{Z}^d}(\sigma_t)_i$ will again be denoted by σ_t for simplicity. Elements of $\Sigma_p^{\mathbb{Z}^d}$ will be denoted by $\bar{\xi}=\bar{\xi}(i,\,j)_{i\in\mathbb{Z}^d,\,j\in\mathbb{Z}^+}$, or $\bar{\xi}=\xi_i(j)_{i\in\mathbb{Z}^d,\,j\in\mathbb{Z}^+}$. For each fixed $i\in\mathbb{Z}^d$, $\xi_i\in\Sigma_p$. This symbolic space is endowed with the distance

$$\rho_q(\bar{\xi},\,\bar{\eta})=\sup_{(i,\,j)\in\mathbb{Z}_+^{d+1}} q^{|i|+|j|}d(\bar{\xi}(i,\,j),\,\bar{\eta}(i,\,j)),$$

where $\mathbb{Z}_+^{d+1} = \{(i, j) : i \in \mathbb{Z}^d, j \in \mathbb{Z}^+\}$ and d is the discrete metric on the set of p symbols. The corresponding metric on $\mathscr{M}$ is the metric ρ_q. It is easy to verify that the map $\bar{\pi}$ is Hölder continuous under the metrics ρ_q.

Since we have proved that the conjugating map h is Hölder continuous in the metric ρ_q, we have the semi-conjugacy $h \circ \bar{\pi}$ between Φ and σ_t. When Φ is a spatial translation invariant perturbation, the conjugating map h is also translation invariant, i.e., $\sigma_s \circ h = h \circ \sigma_s$. Thus, the map $h \circ \bar{\pi}$ is also a semi-conjugacy between the spatial translation σ_s on $\mathscr{M}$ and the spatial translation σ_s on $\Sigma_p^{\mathbb{Z}^d}$. Therefore, $h \circ \bar{\pi}$ is a semi-conjugacy between the $\mathbb{Z}_+^{d+1}$ group actions generated by (Φ, σ_s) and (σ_t, σ_s).

For finite dimensional approximation maps Φ_V, we use the same method to construct symbolic representations through the conjugation map h_V.

The semi-conjugacy acts as a bridge between measures on $\mathscr{M}$ and $\Sigma_p^{\mathbb{Z}^d}$. For a Borel measures μ on $\mathscr{M}$, it has a corresponding measure ν on $\Sigma_p^{\mathbb{Z}^d}$ satisfying the equation

$$\nu((h \circ \bar{\pi})^{-1}(E)) = \mu(E), \quad E \subseteq \mathscr{M}. \tag{4.1}$$

On the other hand, every measure on the symbolic space $\Sigma_p^{\mathbb{Z}^d}$ can be pushed forward to define a measure on $\mathscr{M}$.

4.2. Equilibrium States

We first define equilibrium states for lattice dynamical systems. The description below is adapted from ref. 29.

Let Ω be a compact metric space and τ be a $\mathbb{Z}_+^{d+1}$-action on Ω induced by $d (\geqslant 0)$ commuting homeomorphisms and one continuous map. Let also $\mathscr{U} = \{U_i\}_{i \in I}$ be a cover of Ω: $\bigcup_{i \in I} U_i = \Omega$. For a finite set $\Lambda \subset \mathbb{Z}_+^{d+1}$ define

$$\mathscr{U}^\Lambda = \bigvee_{k \in \Lambda} \tau^{-k} \mathscr{U}$$

to be the refined cover of Ω consisting of all sets of the form

$$B = \bigcap_{k \in \Lambda} \tau^{-k} U_{i(k)}, \quad i(k) \in I.$$

Denote by $|\Lambda|$ the cardinality of the set Λ.

An action τ is said to be *expansive* if there exists $\epsilon > 0$ such that for any $\xi, \eta \in \Omega$,

$$d(\tau^k \xi, \tau^k \eta) \leqslant \epsilon \qquad \text{for all} \quad k \in \mathbb{Z}_+^{d+1} \ \text{implies} \ \xi = \eta.$$

A Borel measure μ on Ω is said to be τ-*invariant* if μ is invariant with respect to all d homeomorphisms and one continuous map. We denote the set of all τ-invariant probability measures on Ω by $\Gamma(\Omega)$.

Let $\mu \in \Gamma(\Omega)$ and $\mathcal{U} = \{U_i\}$ be a finite Borel partition of Ω. Define

$$H(\mu, \mathcal{U}) = -\sum_i \mu(U_i) \log \mu(U_i).$$

and set

$$h_\tau(\mu, \mathcal{U}) = \lim_{a_1,\ldots,a_{d+1} \to \infty} \frac{1}{|\Lambda(a)|} H(\mu, \mathcal{U}^{\Lambda(a)}) = \inf_a \frac{1}{|\Lambda(a)|} H(\mu, \mathcal{U}^{\Lambda(a)}),$$

where $\Lambda(a) = \{(i_1 \ldots i_{d+1}) \in \mathbb{Z}_+^{d+1} : a = (a_1 \ldots a_{d+1}),\ a_n > 0,\ |i_n| \leqslant a_n,\ n = 1,\ldots, d+1\}$. The measure-theoretic *entropy* of the action τ with respect to μ is defined to be

$$h_\mu(\tau) = \sup_{\mathcal{U}} h_\tau(\mu, \mathcal{U}) = \lim_{\operatorname{diam} \mathcal{U} \to 0} h_\tau(\mu, \mathcal{U}), \tag{4.2}$$

where $\operatorname{diam} \mathcal{U} = \max_i (\operatorname{diam} U_i)$.

Let $\mathcal{U}$ be a finite open cover of Ω, φ a continuous function on Ω, and Λ a finite subset of $\mathbb{Z}_+^{d+1}$. Define the partition function over Λ to be

$$Z_\Lambda(\varphi, \mathcal{U}) = \min_{\{B_j\}} \left\{ \sum_j \exp\left[\inf_{\xi \in B_j} \sum_{k \in \Lambda} \varphi(\tau^k \xi) \right] \right\}, \tag{4.3}$$

where the minimum is taken over all subcovers $\{B_j\}$ of $\mathcal{U}^\Lambda$. Set

$$P_\tau(\varphi, \mathcal{U}) = \limsup_{a_1,\ldots,a_{d+1} \to \infty} \frac{1}{|\Lambda(a)|} \log Z_{X(a)}(\varphi, \mathcal{U}).$$

The quantity

$$P_\tau(\varphi) = \lim_{\operatorname{diam} \mathcal{U} \to 0} P_\tau(\varphi, \mathcal{U}) = \sup_{\mathcal{U}} P_\tau(\varphi, \mathcal{U}) \tag{4.4}$$

is called the *topological pressure* of φ. One can show that the limit in expressions (4.2) and (4.4) exists. Details of proofs can be found in ref. 29.

Let $\varphi(x)$ be any continuous function on Ω and $P_\tau(\varphi)$ be its topological pressure with respect to τ. Then, we have the variational principle

$$P_\tau(\varphi) = \sup_{\gamma \in \Gamma} \left(h_\gamma(\tau) + \int \varphi \, d\gamma \right),$$

where $h_\gamma(\tau)$ is the measure theoretical entropy of the $\mathbb{Z}_+^{d+1}$-action τ with respect to γ. A τ-invariant measure μ is called an *equilibrium state* for φ if the supremum is attained at μ.

Equilibrium states exist for continuous functions as long as the $\mathbb{Z}_+^{d+1}$-action is expansive.[29] One can easily rarify that the $\mathbb{Z}_+^{d+1}$-action on $\mathscr{M}$ generated by Φ and the translations is expansive in the metric ρ_q.

The ergodic properties of an equilibrium states are related to its uniqueness. In fact, uniqueness implies ergodicity.[27] The stronger ergodic properties, such as mixing and exponential decay of correlation functions can be obtained by considering symbolic representations of dynamical systems and some other techniques such as the transfer operator method and the zeta-function method.

4.3. Invariant Gibbs States for Symbolic Spaces

From the symbolic representations of Φ and Φ_V, we have the following symbolic spaces (lattice spin systems):

$$\Sigma_p^{\mathbb{Z}^d} = \bigotimes_{i \in \mathbb{Z}^d} (\Sigma_p)_i, \qquad \Sigma_p^V =: \bigotimes_{i \in V} (\Sigma_p)_i. \tag{4.5}$$

Under the metric ρ_q, $0 < q < 1$, $\Sigma_p^{\mathbb{Z}^d}$ and Σ_p^V are compact metric spaces and the variational principle holds in these cases. For spaces Σ_p^V, we have a $\mathbb{Z}_+$-action induced by the left shift σ_t of Σ_p. On $\Sigma_p^{\mathbb{Z}^d}$, we have a $\mathbb{Z}_+^{d+1}$-action induced by the (space) translations σ_s on $\mathbb{Z}^d$ and (the time shift) σ_t on Σ_p. Clearly, both actions are expansive in the metric ρ_q.

On these symbolic spaces, equilibrium states for any Hölder continuous function are equivalent to invariant *Gibbs states* (defined below; also see chapter one and three of ref. 29). Even though the equivalence theorem there was proved for $\mathbb{Z}^d$-actions, the proofs are valid for $\mathbb{Z}_+^{d+1}$-actions).

Any element $\bar{\xi} \in \Sigma_p^{\mathbb{Z}^d}$ will also be called a *configuration*. For any subset $\Lambda \subset \mathbb{Z}_+^{d+1}$, set

$$\Omega_\Lambda = \{1, 2,..., p\}^\Lambda.$$

For convenience, elements of Ω_Λ are also denoted by ξ_Λ, or $\xi(\Lambda)$. One can say that Ω_Λ consists of restrictions of configurations $\bar{\xi}$ to Λ.

For each finite subset $\Lambda \subset \mathbb{Z}_+^{d+1}$, define a function $p_\Lambda(\bar{\xi})$ on $\Sigma_\Lambda^{\mathbb{Z}^d}$ by

$$p_\Lambda(\bar{\xi}) = \frac{1}{\sum_{\bar{\eta}\,:\,\eta(\hat{\Lambda}) = \xi(\hat{\Lambda})} \exp(\sum_{k \in \mathbb{Z}_+^{d+1}} \varphi(\tau^k \bar{\eta}) - \varphi(\tau^k \bar{\xi}))}, \tag{4.6}$$

where τ^k denotes the action $\sigma_s^i \circ \sigma_t^j$, $\hat{\Lambda} = \mathbb{Z}_+^{d+1} \setminus \Lambda$ and $k = (i, j)$, $i \in \mathbb{Z}^d$, $j \in \mathbb{Z}^+$.

Let φ be a Hölder continuous function on $\Sigma_p^{\mathbb{Z}^d}$. A probability measure μ on $\Sigma_p^{\mathbb{Z}^d}$ is called a Gibbs state for φ if for any finite subset $\Lambda \subset \mathbb{Z}_+^{d+1}$,

$$\mu_\Lambda(\xi(\Lambda)) = \int_{\Omega_{\hat{\Lambda}}} p_\Lambda(\bar{\xi})\, d\mu_{\hat{\Lambda}}, \tag{4.7}$$

where μ_Λ and $\mu_{\hat{\Lambda}}$ are projections of μ onto Ω_Λ and $\Omega_{\hat{\Lambda}}$, respectively. Equation (4.7) is known as the Dobrushin-Ruelle-Lanford equation.

There are other equivalent ways to define Gibbs states for Hölder continuous functions on symbolic spaces. Let φ be such a function. For each finite volume Λ, we first define a *conditional Gibbs distribution* on Ω_Λ under a given boundary condition η^* by

$$\mu_{\eta^*, \Lambda}(\xi(\lambda)) = \frac{1}{\sum_{\eta,\, \eta(\hat{\Lambda}) = \eta^*(\hat{\Lambda})} \exp(\sum_{k \in \mathbb{Z}^{d+1}} \varphi(\tau^k \eta) - \varphi(\tau^k(\xi(\Lambda) + \eta^*(\hat{\Lambda})))}, \tag{4.8}$$

where $\xi(\Lambda) + \eta^*(\hat{\Lambda})$ denotes the configuration on $\Lambda \cup \hat{\Lambda}$ whose restrictions to Λ and $\hat{\Lambda}$ are $\xi(\Lambda)$ and $\eta^*(\hat{\Lambda})$ respectively. Then the set of all Gibbs states for φ is the convex hull of thermodynamic limits of the conditional Gibbs distributions.[29]

In order to establish the correspondence between equilibrium states of lattice dynamical systems and the invariant Gibbs states of lattice spin systems, we will need ergodic properties of Gibbs states that is related to the uniqueness of Gibbs states. The uniqueness of Gibbs states for various potential functions have been major research topics in equilibrium statistical mechanics during last three decades. It is well-known that Gibbs states are always unique for any Hölder continuous functions on one dimensional lattice spin systems (so called absence of phase transition). In higher dimensional cases, the uniqueness holds for those Hölder functions with a small Hölder constant which corresponds to the situation of "high temperature." For general Hölder continuous functions, the uniqueness is not true. The Ising model provides a simple example.[20]

The potential function that will appear in our consideration of SRB-measures for lattice dynamical systems in the next section does not have a small Hölder constant. However, the potential function is only a small perturbation from a potential function for which the uniqueness holds. Using a direct cluster expansion technique one can show that the same properties hold for the slightly perturbed potential functions. We state the theorem below. The theorem was proved in ref. 19 for the dimension two case ($d = 1$) for a class of subshifts of finite type. The proof provides a formula for the Gibbs state in terms of the potential. For general higher dimensional cases, it was proved in refs. 7 and 8. The latter shows directly the uniqueness without obtaining an explicit expression of the Gibbs state (see ref. 7).

Theorem 5 (Uniqueness and Exponential Mixing Property of Gibbs States). Let φ be a Hölder continuous function on $\Sigma_p^{\mathbb{Z}^d}$. Assume that φ can be written in the form $\varphi = \varphi_0 + \varphi_1$, where φ_0 is a Hölder continuous function satisfying the condition $\varphi_0(\bar{\xi}) = \varphi_0(\bar{\eta})$ for all $\bar{\xi}, \bar{\eta} \in \Sigma_p^{\mathbb{Z}^d}$ with $\xi(0, j) = \eta(0, j)$, $0 \in \mathbb{Z}^d$, $j \in \mathbb{Z}^+$ and φ_1 satisfies the condition $|\varphi_1(\bar{\xi}) - \varphi_1(\bar{\eta})| \leqslant c\rho_q^\delta(\bar{\xi}, \bar{\eta})$ with the Hölder coefficient c sufficiently small. Then, the Gibbs state for $\varphi = \varphi_0 + \varphi_1$ is unique and exponentially mixing with respect to the $\mathbb{Z}_+^{d+1}$-action.

4.4. Semi-Conjugacy

Now we are ready to construct equilibrium states on lattice dynamical systems that correspond to invariant Gibbs states on their symbolic representations. First we prove the following lemma on the transition of potential functions. Note that invariant Gibbs states and equilibrium states are the same on our symbolic space $\Sigma_p^{\mathbb{Z}^d}$ (ref. 29, p. 60).

Lemma 4. Let φ_0 and φ_1 be Hölder continuous function on $\mathscr{M}$ satisfying the condition $\varphi_0(\bar{x}) = \varphi_0(\bar{y})$ whenever $x_0 = y_0$. Then, for every $\delta > 0$ and $0 < q < 1$, there exist sufficiently small $c' > 0$ and $\epsilon > 0$ such that when $|\varphi_1(\bar{x}) - \varphi_1(\bar{y})| \leqslant c'\rho_q^\delta(\bar{x}, \bar{y})$, the composition of functions $(\varphi_0 + \varphi_1)(h \circ \bar{\pi})$ satisfies the condition of Theorem 5 for suitably chosen constants. Consequently, the invariant Gibbs state (or the equilibrium state) on $\Sigma_p^{\mathbb{Z}^d}$ for $(\varphi_0 + \varphi_1)(h \circ \bar{\pi})$ is unique and exponentially mixing with respect to the $\mathbb{Z}_+^{d+1}$-action.

Proof. We need to show that $(\varphi_0 + \varphi_1)(h \circ \bar{\pi}(\bar{\xi}))$ satisfies the condition of Theorem 5. It suffices to show that $\varphi_0(h \circ \bar{\pi})$ can be written into the form $\varphi_0(h \circ \bar{\pi}) = \psi_0(\bar{\xi}) + \psi_1(\bar{\xi})$ with ψ_0 and ψ_1 satisfying the conditions of Theorem 5.

Pick any fixed configuration $\bar{\xi}^* \in \Sigma_p^{\mathbb{Z}^d}$. Denote by $(\xi_0, \bar{\xi}^*)$ the configuration whose restriction to the lattice site $0 \in \mathbb{Z}^d$ is the same as that of $\bar{\xi}$ and whose values elsewhere are the same as those of $\bar{\xi}^*$. Define

$$\psi_0(\bar{\xi}) = \varphi_0(h \circ \bar{\pi}(\xi_0, \bar{\xi}^*));$$

and

$$\psi_1(\bar{\xi}) = \varphi_0(h \circ \bar{\pi}(\bar{\xi})) - \varphi_0(h \circ \bar{\pi}(\xi_0, \bar{\xi}^*)).$$

Notice that the value of ψ_0 depends only on ξ_0. Therefore, we need only to verify that ψ_1 is Hölder continuous with some exponent and a coefficient that can be made arbitrary small as c' and ϵ are small.

Let us pick two configurations $\bar{\xi}, \bar{\eta} \in \Sigma_p^{\mathbb{Z}^d}$. It suffices to prove the following inequality

$$|\psi_1(\bar{\xi}) - \psi_1(\bar{\eta})| \leqslant c_0(\epsilon)\, q'^{|i|+|j|}$$

for some constant $0 < q' < 1$ and all $\bar{\xi}, \bar{\eta}$ with $\xi_k(l) = \eta_k(l)$ for every $(k, l) \in \mathbb{Z}_+^{d+1}$ except at the site (i, j).

Let n_0 be a large integer. If $|i| + |j| \leqslant n_0$, we have

$$\begin{aligned}
&|\psi_1(\bar{\xi}) - \psi_1(\bar{\eta})| \\
&\quad \leqslant |\varphi_0(h \circ \bar{\pi}(\bar{\xi})) - \varphi_0(h \circ \bar{\pi}(\xi_0, \bar{\xi}^*))| + |\varphi_0(h \circ \bar{\pi}(\bar{\eta})) - \varphi_0(h \circ \bar{\pi}(\eta_0, \bar{\xi}^*))| \\
&\quad \leqslant L_1 \rho_q^{\alpha_1}(h \circ \bar{\pi}(\bar{\xi}), h \circ \bar{\pi}(\xi_0, \bar{\xi}^*)) + L_1 \rho_q^{\alpha_1}(h \circ \bar{\pi}(\eta), h \circ \bar{\pi}(\eta_0, \bar{\xi}^*)) \\
&\quad \leqslant L_1 c^{\alpha_1}(\epsilon)\, L_2^{\alpha_1\alpha_2} + L_1 c^{\alpha_1}(\epsilon)\, L_2^{\alpha_1\alpha_2} \\
&\quad \leqslant \frac{2L_1 c^{\alpha_1}(\epsilon)\, L_2^{\alpha_1\alpha_2}}{q^{n_o}}\, q^{|i|+|j|},
\end{aligned}$$

where L_1, α_1 are the Hölder coefficient and exponent of φ_0, L_2, α_2 are the Hölder coefficient and exponent of $\bar{\pi}$, and $c(\epsilon)$ is the constant from Theorem 4.

If $|i| + |j| > n_0$, we have

$$\begin{aligned}
|\psi_1(\bar{\xi}) - \psi_1(\bar{\eta})| &\leqslant |\varphi_0(h \circ \bar{\pi}(\bar{\eta})) - \varphi_0(h \circ \bar{\pi}(\bar{\xi}))| \\
&\quad + |\varphi_0(h \circ \bar{\pi}(\eta_0, \bar{\xi}^*)) - \varphi_0(h \circ \bar{\pi}(\xi_0, \bar{\xi}^*))| \\
&\leqslant 2L_1 C^{\alpha_1} L_2^{\alpha_1\alpha_2} q^{\alpha_1\alpha_2(|i|+|j|)} \leqslant 2L_1 C^{\alpha_1} L_2^{\alpha_1\alpha_2} \left(\frac{q^{\alpha_1\alpha_2}}{q'}\right)^{n_0} q'^{|i|+|j|}.
\end{aligned}$$

Let

$$c_0(\epsilon) = \max\left\{\frac{2L_1 c^{\alpha_1}(\epsilon)\, L_2^{\alpha_1\alpha_2}}{q^{n_o}}, 2L_1 C^{\alpha_1} L_2^{\alpha_1\alpha_2} \left(\frac{q^{\alpha_1\alpha_2}}{q'}\right)^{n_0}\right\}.$$

It can be made arbitrarily small when we choose q' such that $q^{\alpha_1\alpha_2} < q' < 1$ and ϵ small. ∎

The following theorem summarizes the connections between equilibrium states of lattice dynamical systems (Φ, σ_s) on $\mathcal{M}$ and the invariant Gibbs states on their symbolic representations.

Theorem 6. For any Hölder continuous function $\varphi_0(\bar{x})$ on $\mathcal{M}$ that depends only on the coordinate x_0, there exist $\epsilon_0 > 0$, $c_0 > 0$ such that when $\epsilon \leqslant \epsilon_0$ and φ_1 is a Hölder continuous function with a Hölder coefficient smaller than c_0, the follow statements hold.

(1) The invariant Gibbs state ν for the function $(\varphi_0 + \varphi_1)(h \circ \bar{\pi})$ on $\Sigma_p^{\mathbb{Z}^d}$ is unique and exponentially mixing with respect to the $\mathbb{Z}_+^{d+1}$-action (σ_t, σ_s).

(2) The measure μ on $\mathcal{M}$ defined by $\mu(E) = \nu((h\pi)^{-1}(E))$ is invariant under the $\mathbb{Z}_+^{d+1}$-action generated by (Φ, σ_s) and is the unique equilibrium state satisfying the *variational principle*

$$P_\tau(\varphi_0 + \varphi_1) = h_\mu(\tau) + \int (\varphi_0 + \varphi_1)\, d\mu.$$

Moreover, the measure μ is exponentially mixing with respect to the $\mathbb{Z}_+^{d+1}$-action generated by (Φ, σ_s).

Remark 3. The role of ϵ_0 is to control the conjugating map h so that the composition $(\varphi_0 + \varphi_1)(h \circ \bar{\pi})$ satisfies the condition of Theorem 5. The proof of the statement (1) follows directly from the lemma. To prove the statement (2), we need to use the Hölder continuity of h, the ergodicity of the Gibbs state which is guaranteed by its uniqueness, and the fact that h is "almost" a homeomorphism. The proof follows the standard technique in ref. 3. Details were presented in ref. 15.

5. SRB-MEASURES FOR LATTICE DYNAMICAL SYSTEMS

We now focus on the SRB-measure for lattice dynamical systems. The existence, uniqueness, and exponential mixing property of SRB measures for coupled map lattices were proved in ref. 7 with the transfer operator technique under essentially the same conditions. Here, we prove that this measure is an equilibrium state satisfying the variational principle with respect to the $\mathbb{Z}_+^{d+1}$-action generated by (Φ, σ_s). Recent progress concerning the uniqueness and limit theorems of the SRB measure and properties of transfer operators can be found in refs. 2, 10, 18, 24, and 30.

The construction of SRB-measures for lattice systems is based on the finite dimensional approximation. We will show that the SRB-measures μ_V on the finite dimensional space $M_V = \bigotimes_{i \in V} M_i$ converges to a measure μ on $\mathscr{M}$ in the sense of thermodynamic limit. We observe that measures μ_V are not supported on the same space. For $V \subset V' \subset \mathbb{Z}^d$, the projection of $\mu_{V'}$ onto M_V is a probability measure. The convergence is understood in the following sense: for every $V \subset \mathbb{Z}^d$, the projection of $\mu_{V'}$ on M_V weak* converges to the projection of μ onto M_V as $V' \to \mathbb{Z}^d$. We will show that this measure μ is a unique equilibrium state under $\mathbb{Z}_+^{d+1}$-action for a Hölder continuous function φ satisfying the condition of Theorem 6. This Hölder continuous potential function is a small perturbation of $-\log |f'(x_0)|$. Thus, μ is exponentially mixing with respect to (Φ, σ_s). The approach of the proof is to consider corresponding Gibbs states on lattice spin systems.

5.1. Limit of SRB-Measures

We consider the thermodynamic limit of the sequence of SRB measures μ_V for expanding maps Φ_V on M_V as $V \to \mathbb{Z}^d$. We require that the perturbation Φ satisfies additional condition (C5), which is never used in the previous sections.

We first state the main theorem and the strategy of the proof. Denote $\nu_V = (h_V \bar{\pi})^{-1} \mu_V$ the pull back measure on the symbolic space $\bigotimes_{i \in V} \Sigma_p$. This measure is the unique Gibbs state for the potential function $-\log J\Phi_V(h_V \bar{\pi})$ with respect to the $\mathbb{Z}_+$-action σ_t.

Theorem 7. Under the conditions (C1)–(C5) for sufficiently small $\epsilon > 0$ and the assumption that the perturbation is spatial translation invariant,

(1) the measure ν_V converges to a measure ν on $\Sigma_p^{\mathbb{Z}^d}$. The measure ν is invariant under the $\mathbb{Z}_+^{d+1}$-action generated by (σ_t, σ_s) and it is the unique and exponentially mixing Gibbs state for some potential function $\varphi(h_V \circ \bar{\pi})$ close to $-\log |f'(x_0)|$;

(2) the push-forward measure $\mu = (h \circ \bar{\pi})^* \nu$ is the unique equilibrium state for the potential function $\varphi(\bar{x})$. The measure μ is exponentially mixing with respect to the $\mathbb{Z}_+^{d+1}$-action τ generated by (Φ, S). Moreover, the entropy formula holds:

$$h_\mu(\tau) = \int \varphi \, d\mu.$$

The proof of the theorem consists of a careful decomposition (or localization) of the potential function $-\log J\Phi_V(h_V \bar{\pi})$. The technique was used in ref. 7 and later presented in full detail in ref. 20 for coupled hyperbolic maps. Since our local map f is an expanding map on the circle, the calculation becomes more transparent and a lot of technical difficulties related to the regularity of stable and unstable manifolds can be avoided. In fact, in certain cases (nearest-neighbor interaction), the expression of the potential function φ can be explicitly calculated in terms of coupling strength and other parameters.

5.2. The Decomposition of $-\log J\Phi_V$ and the Construction of the Potential Function φ

We shall arrange the elements of $V \subset \mathbb{Z}^d$ in certain linear order and denote the total number of elements in V by $|V|$.

We rewrite the derivative matrix $D\Phi_V$ in the following

$$D\Phi_V = \left(\frac{\partial \Phi_i}{\partial x_j}\right)_{i,\, j \in V} = (\mathrm{diag}(f'(x_i))(I + A_V(x_V)),$$

where $(\mathrm{diag}(f'(x_i))$ denotes the diagonal matrix with $\{f'(x_i)\}$ on the main diagonal.

Under conditions (C1)–(C5), the entries of the matrix $A_V(x_V)$, $a_{ij}(x_V)$, $i, j \in V$ have the following properties.

Lemma 5

(1) $|a_{ij}(x_V))| \leqslant \epsilon C_3 e^{-\beta |i-j|}$.

(2) $|a_{ij}(x_V) - a_{ij}(y_V)| \leqslant \epsilon C_4 e^{-\beta |i-k|} d^\alpha(x_k, y_k)$ for any x_V, y_V with $x_l = y_l$, $l \in V, l \neq k$.

(3) For any $V \subset V'$, $i, j \in V$, $a_{ij}(x_V) = a_{ij}(x_V, x^*_{V' \setminus V})$.

(4) $|a_{ij}(x_V) - a_{ij}(y_{V'})| \leqslant \epsilon C_5 e^{-\frac{\beta}{2} d(i, \partial V)}$,

where C_5 is a constant, $V \subset V' \subset \mathbb{Z}^d$, $x_l = y_l$, $l \in V$, and $d(i, \partial V)$ denotes the distance between i and the boundary of V in $\mathbb{Z}^d$.

Proof. All these properties are direct consequences of our definition of the perturbation. (1) comes from condition (C4) and (2) comes from condition (C5). (3) is from the definition of Φ_V while (4) is a consequence of (2) and (3). ∎

Next, we use the following formula to expand of a determinant of a matrix B

$$\det(\exp(B)) = \exp(\mathrm{trace}(B)).$$

In our context, $\exp(B) = I + A_V(x_V)$, or $B = \ln(I + A_V)$. Then,

$$\det(I + A_V) = \exp(\text{trace}(\ln(I + A_V)) = \exp\left(-\sum_{i \in V} w_{Vi}\right),$$

where

$$w_{Vi}(x_V) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} a_{ii}^{(n)}(x_V) \tag{5.1}$$

and $a_{ii}^{(n)}(x_V)$ are entries on the main diagonal of $(A_V)^n$. Thus, we have

$$J\Phi_V = \exp\left(-\sum_{i \in V} (-\log |f'(x_i)| + w_{Vi})\right) \tag{5.2}$$

Lemma 6. The functions $w_{Vi}(x_V)$ satisfy the following properties.

(1) $|w_{Vi}(x_V)| \leqslant C\epsilon$.

(2) $|w_{Vi}(x_V) - w_{Vi}(y_V)| \leqslant C\epsilon \exp(-\frac{\beta}{2}|i-k|)\, d^{\delta}(x_k, y_k)$, for any x_V, y_V with $x_l = y_l$, $l \in V$, $l \neq k$.

(3) For $V \subset V'$, $|w_{Vi}(x_V) - w_{V'i}(x_V, y_{V' \setminus V})| \leqslant C\epsilon \exp(-\frac{\beta}{2} d(i, \partial V))$.

(4) $\varphi_i(\bar{x}) = \lim_{V \to \mathbb{Z}^d} w_{Vi}(x_V)$ exists and is translation invariant in the following sense: $\varphi_i(\bar{x}) = \varphi_0(\sigma_s^i \bar{x})$.

Proof. The proof consists of straightforward computations. We first show the following estimation

$$|a_{ij}^{(n)}| \leqslant (C\epsilon)^n\, e^{-\tilde{\beta}\,|i-j|}, \tag{5.3}$$

where $\tilde{\beta}$ is any number smaller than β and $C = C(\tilde{\beta})$ is a constant.

We use induction. For $n = 2$, we have

$$\begin{aligned}
|a_{ij}^{(2)}| &= \left| \sum_{l \in V} a_{il} a_{lj} \right| \leqslant \sum_{l \in V} \epsilon^2 C_3^2 \exp(-\beta(|i-l| + |l-j|)) \\
&\leqslant \sum_{l \in V} \epsilon^2 C_3^2 \exp(-\tilde{\beta}(|i-l| + |l-j|) - (\beta - \tilde{\beta})\,|l-j|) \\
&\leqslant \epsilon^2 C_3^2 e^{-\tilde{\beta}\,|i-j|} \sum_{l \in V} \exp(-(\beta - \tilde{\beta})\,|l-j|) \leqslant C_3 C \epsilon^2 e^{-\tilde{\beta}\,|i-j|}, \\
&\leqslant C^2 \epsilon^2 e^{-\tilde{\beta}\,|i-j|},
\end{aligned} \tag{5.4}$$

where $C = C(\tilde{\beta}) = \sum_{l \in \mathbb{Z}^d} \exp(-(\beta - \tilde{\beta})\,|l|)\, C_3$.

Let us assume that $|a_{ij}^{(n-1)}| \leqslant C^{n-1}\epsilon^{n-1} \exp(-\tilde{\beta}\,|i-j|)$. Then

$$|a_{ij}^{(n)}| = \left|\sum_{l \in V} a_{il}^{(n-1)} a_{lj}\right| \leqslant \sum_{l \in V} C^{n-1}\epsilon^n C_3 \exp(-\tilde{\beta}(|i-l|+|l-j|)-(\beta-\tilde{\beta})\,|l-j|)$$
$$\leqslant C^n\epsilon^n \exp(-\tilde{\beta}\,|i-j|). \tag{5.5}$$

Therefore (1) follows directly from the definition of w_{Vi} with another different constant C.

To prove (2), we need only to show the following estimation:

$$|a_{ij}^{(n)}(x_V)-a_{ij}^{(n)}(y_V)| \leqslant (C\epsilon)^n\, e^{-\frac{\beta}{2}|i-k|} d^{\alpha}(x_k, y_k),$$

for any x_V, y_V with $x_l = y_l$, $l \in V$, $l \neq k$. We again use induction. For $n = 2$,

$$|a_{ij}^{(2)}(x_V)-a_{ij}^{(2)}(y_V)|$$
$$= \left|\sum_{l \in V} a_{il}(x_V)\, a_{lj}(x_V)-a_{il}(y_V)\, a_{lj}(y_V)\right|$$
$$= \left|\sum_{l \in V} a_{il}(x_V)[a_{lj}(x_V)-a_{lj}(y_V)]+a_{lj}(y_V)[a_{il}(x_V)-a_{il}(y_V)]\right|$$
$$\leqslant \sum_{l \in V} \epsilon^2 C_3 C_4[\exp(-\beta(|l-k|+|i-l|))+\exp(-\beta(|l-j|+|i-k|))]\, d^{\alpha}(x_k, y_k)$$
$$\leqslant C\epsilon^2 \exp\left(-\frac{\beta}{2}\,|i-k|\right) d^{\alpha}(x_k, y_k),$$

where $C = 2C_3C_4 \sum_{l \in \mathbb{Z}^d} \exp(-\frac{\beta}{2}\,|l|)$.

For general n, we estimate similarly using the estimations Lemma 5 (1) and (2).

$$|a_{ij}^{(n)}(x_V)-a_{ij}^{(n)}(y_V)|$$
$$= \left|\sum_{l \in V} a_{il}^{(n-1)}(x_V)\, a_{lj}(x_V)-a_{il}^{(n-1)}(y_V)\, a_{lj}(y_V)\right|$$
$$= \left|\sum_{l \in V} a_{il}^{(n-1)}(x_V)[a_{lj}(x_V)-a_{lj}(y_V)]+a_{lj}(y_V)[a_{il}^{(n-1)}(x_V)-a_{il}^{(n-1)}(y_V)]\right|$$
$$\leqslant \sum_{l \in V} (C\epsilon)^{n-1}\, \epsilon C_3 C_4 \left[\exp\left(-\frac{\beta}{2}\,|i-l|\,-\beta\,|l-k|\right)\right.$$
$$\left.+\exp\left(-\beta\,|l-j|-\frac{\beta}{2}\,|i-k|\right)\right] d^{\alpha}(x_k, y_k)$$
$$\leqslant (C\epsilon)^n \exp\left(-\frac{\beta}{2}\,|i-k|\right) d^{\alpha}(x_k, y_k).$$

Statement (3) is proved similarly by using the corresponding property (4) in Lemma 5 for the matrix A_V. Property (4) comes from (3) and our assumption that map Φ is spatial translation invariant. Note that the convergence is uniform in $\bar{x}$. ∎

We set $\psi(\bar{x}) = \varphi_0(\bar{x})$. A straightforward calculation shows that Lemma 6 (2) implies that $\psi(\bar{x})$ is a Hölder continuous function with a small Hölder constant. In fact, the Hölder constant goes to zero as ϵ goes to zero. Thus, by Theorem 6 the equilibrium state for $\varphi = \psi(\bar{x}) - \log|f'(x_0)|$ with respect to the $\mathbb{Z}_+^{d+1}$-action generated by (Φ, σ_s) is unique and exponentially mixing.

Theorem 8. The Gibbs states ν_V for potentials $\varphi_V(\xi_V) = -\log J\Phi_V(h_V \pi_V(\xi_V))$ on the one dimensional lattice spin systems Σ_p^V converge to a Gibbs state on the $(d+1)$-dimensional lattice spin system $\Sigma_p^{\mathbb{Z}^d}$. This Gibbs state is uniquely determined by the Hölder continuous potential function $\varphi(h\bar{\pi}(\bar{\xi})) = \psi(h\bar{\pi}(\bar{\xi})) - \log|f'((h\bar{\pi}(\bar{\xi}))_0)|$ and is exponentially mixing respect to the $\mathbb{Z}_+^{d+1}$-action of the lattice.

Theorem 9. The SRB measure μ_V for Φ_V converges to an equilibrium measure on the space $\mathcal{M}$ as $V \to \mathbb{Z}^d$. This equilibrium measure is uniquely determined by a Hölder continuous potential function $\psi(\bar{x}) - \log|f'(x_0)|$ defined on $\mathcal{M}$ and the measure is exponentially mixing with respect to both spatial translations and Φ. The function $\psi(\bar{x})$ is given by the formula

$$\psi(\bar{x}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} a_{00}^{(n)}(\bar{x}),$$

where $a_{00}^{(n)}(\bar{x})$ is the entry of the infinite matrix A^n corresponding to the $(0, 0)$ lattice point of $\mathbb{Z}^d \times \mathbb{Z}^d$ and the matrix $A(\bar{x})$ is defined by the relation

$$\left(\frac{\partial \Phi_i}{\partial x_j}(\bar{x})\right)_{i, j \in \mathbb{Z}^d} = (\mathrm{diag}(f'(x_i)))(I + A(\bar{x})).$$

Remark 4. We give a sketch of the idea with which the potential function $\varphi(h\bar{\pi}(\bar{\xi})) = \psi(h\bar{\pi}(\bar{\xi})) - \log f'((h\bar{\pi}(\bar{\xi}))_0)$ is obtained. We need to decompose the Hamiltonian of the Gibbs state ν_V with respect to the time shift σ_t

$$\sum_{j \in \mathbb{Z}^+} -\log J\Phi_V(h_V \pi_V(\sigma_t^j \xi_V))$$

to obtain the Hamiltonian for the Gibbs state ν with respect to the $\mathbb{Z}_+^d$-action $(\sigma_s^i \circ \sigma_t^j)$. Using the expression (5.2), we have

$$\sum_{j\in\mathbb{Z}^+} -\log J\Phi_V(h_V\pi_V(\sigma_t^j\xi_V)) \sum_{j\in\mathbb{Z}^+}\sum_{i\in V} -\log|f'(h_V\pi_V(\sigma_s^i\sigma_t^j\xi_V))|+w_{Vi}(h_V\pi_V(\sigma_t^j\xi_V))$$

$$\to \sum_{j\in\mathbb{Z}^+, i\in\mathbb{Z}^d} [-\log|f'(x_i)|+\varphi_0](h\pi(\sigma_s^i\sigma_t^j\bar{\xi})).$$

The actual proof uses the equivalent description of Gibbs states with conditional Gibbs distributions (4.8). Details were presented in ref. 20.

Theorem 9 follows from Theorem 8 by using the semi-conjugacy.

Remark 5. The entropy formula in Theorem 7 and the decomposition of the potential function $-\log J\Phi_V$ for the SRB measure of Φ_V have an interesting consequence on the relation between the entropy $h_\mu(\tau)$ of the lattice system and the entropy $h_{\mu_V}(\Phi_V)$. Since $\mu_V \to \mu$ weakly and $w_{Vi}(x_V)$ converges to $\varphi_i(\bar{x})$ uniformly by Lemma 6, we have

$$h_\mu(\tau) = \int_{\mathcal{M}} \varphi \, d\mu = \lim_{V\to\mathbb{Z}^d} \frac{1}{|V|} \int_{\mathcal{M}} -\log J\Phi_V \, d\mu_V = \lim_{V\to\mathbb{Z}^d} \frac{1}{|V|} h_{\mu_V}(\Phi_V).$$

6. THE POTENTIAL FUNCTION AND THE ENTROPY OF COUPLED MAP LATTICE

In this section, we go one step further to determine explicit formulas of potential functions for coupled expanding maps (CMLs) on the circle. As a consequence, we obtain the formula of the (spatiotemporal) entropy of the coupled map lattice when the interaction is of nearest neighbor type.

6.1. The Potential Function φ

To obtain the potential function for the SRB measure of coupled map lattices, we first calculate the Jacobian matrix of the map $\Phi = G \circ F$:

$$D\Phi(\bar{x}) = DG(F(\bar{x}))\, DF(\bar{x}).$$

Notice that both $DG = (\frac{\partial G_i}{\partial x_j}(F(\bar{x})))$ and $DF = (f'(x_i))$ are infinite matrices indexed by (i, j), $i, j \in \mathbb{Z}^d$ and translation invariant since we assume that G is translation invariant. We write DG in the following form:

$$DG = \left(\frac{\partial G_i}{\partial x_i} F(\bar{x})\right)(I + A),$$

where $(\frac{\partial G_i}{\partial x_i}(F(\bar{x}))$ denotes the diagonal matrix with $\frac{\partial G_i}{\partial x_i}(F(\bar{x}))$ on the main diagonal. Thus, for $i \neq j$, a_{ij}, the entries off diagonal of A are given by the formula

$$a_{ij} = \frac{\partial G_i}{\partial x_j}(F(\bar{x})) / \frac{\partial G_i}{\partial x_i}(F(\bar{x}))$$

and $a_{ii} = 0$.

By the expansion (5.2) and Remark 4, we see that the potential function for the SRB-measure with respect to the $\mathbb{Z}_+^{d+1}$-action on $\mathscr{M}$ induced by the map Φ and d translations is

$$\varphi(\bar{x}) = -\log |f'(x_0)| - \log \left| \frac{\partial G_0}{\partial x_0}(F(\bar{x})) \right| + \psi^*(F(\bar{x})), \tag{6.1}$$

where

$$\psi^*(F(\bar{x})) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} a_{00}^{(n)}(F(\bar{x})) \tag{6.2}$$

and $a_{00}^{(n)}(F(\bar{x}))$ is the entry at the lattice site $0 \times 0 \in \mathbb{Z}^d \times \mathbb{Z}^d$ of the infinite matrix $A^n = AA \cdots A$. The term $-\log \frac{\partial G_0}{\partial x_0}(F(\bar{x}))$ is isolated out from the term $\psi(\bar{x})$ in Theorem 9 in the potential function corresponding to the perturbation. It is for the convenience of later calculation.

6.2. Potential Functions for CMLs with Nearest Neighbor Interactions

The infinite size of the matrix A poses a special difficulty for further calculation of the potential function in terms of the coupled map. In the simple situation of the nearest neighbor interaction, however, the calculation can be directly carried out using a standard technique in statistical physics.

6.2.1. The Lattice $\mathbb{Z}^1$ Case

We first assume that $d = 1$ and the perturbation map G has the following form:

$$G = (G_i): \quad G_i(\bar{x}) = g(x_{i-1}, x_i, x_{i+1})$$

for some differentiable function $g(x, y, z)$.

The infinite matrix A formulated in the previous section can be expressed as a sum of two matrices: $A = L + R$, where $L = (l_{ij})$ is an infinite matrix with the property

$$l_{ij} = 0, \quad j \neq i-1, \quad l_{ii-1} = \frac{\partial G_i}{\partial x_{i-1}} \Big/ \frac{\partial G_i}{\partial x_i},$$

i.e., the weighted leftward shift operator,and $R = (r_{ij})$ is an infinite matrix with the property

$$r_{ij} = 0, \quad j \neq i+1, \quad r_{ii+1} = \frac{\partial G_i}{\partial x_{i+1}} \Big/ \frac{\partial G_i}{\partial x_i},$$

i.e., the weighted rightward shift operator.

For convenience, we denote $\alpha_i = l_{ii-1}$ and $\beta_i = r_{i-1i}$. Note that the product matrix LR is a diagonal matrix with the entry at (i, i) being $\alpha_i \beta_i$. We denote this diagonal matrix by T. We will determine the function ψ^* in terms of α_i and β_i's. Let Π denote the collection of all sequences of two symbols L and R of length $n = 2k$ with equal numbers of L's and R's, i.e.,

$$\Pi = \{\pi = (C_1 C_2 \cdots C_{2k}) : C_l = L \text{ or } R, 0 \leqslant l \leqslant 2k \text{ and the number of } L\text{'s is } k\}.$$

Proposition 2

$$a_{00}^{(n)} = \begin{cases} 0 & \text{if} \quad n \text{ is odd}, \\ \sum_{\pi \in \Pi} \alpha_{m_1} \beta_{m_1} \alpha_{m_2} \beta_{m_2} \cdots \alpha_{m_k} \beta_{m_k} & \text{if} \quad n = 2k, \end{cases}$$

where the sequence of integers $(m_1, m_2,..., m_k)$ is determined by each π.

Proof. We first introduce a linear operator on infinite matrices: the shift along the diagonal:

If $A = (a_{ij})$ is an infinite matrix, then A^s denotes the matrix whose entry at (i, j) is $a_{i+1, j+1}$. We also denote $(A^s)^s = A^{s^2}$ and $A = A^{s^0} = (A^s)^{s^{-1}}$. i.e., we have a $\mathbb{Z}$-action on infinite matrices.

With the help of this $\mathbb{Z}$-action, we can expand $A^n = (L+R)^n$ in terms of a_i's and b_i's.

We observe that

(1) $RL = T^s$.

(2) If B is any diagonal matrix, $BR = RB^s$ and $BL = LB^{s^{-1}}$.

Let P be a product of n copies of either L or R's such as $LRRRL\cdots R$. Each product corresponds to an element π in the direct product space of two symbols over n places. Assume that we have k L matrices and $n-k$ R matrices. Then, using the properties (1) and (2), we have

$$P=\begin{cases} L^{2k-n}T^{s^{m_1}}T^{s^{m_2}}\cdots T^{s^{m_{n-k}}}, & \text{if} \quad k>n-k;\\ R^{n-2k}T^{s^{m_1}}T^{s^{m_2}}\cdots T^{s^{m_k}}, & \text{if} \quad k<n-k;\\ T^{s^{m_1}}T^{s^{m_2}}\cdots T^{s^{m_k}}, & \text{if} \quad k=n-k, \end{cases}$$

where the sequence of integers $(m_1,\ldots, m_k)$ is determined by each element π.

For example,

$$RLRLRLRLRRR=T^sT^sT^sRRR=RT^{s^2}T^{s^2}T^{s^2}RR=\cdots=R^3T^{s^4}T^{s^4}T^{s^4},$$

and

$$LLLRLLR=L^3T^sT.$$

Since the product $T^{s^{m_1}}T^{s^{m_2}}\cdots T^{s^{m_k}}$ is a diagonal matrix, we have that all entries on the diagonal of P are zero except in the case of $k=n-k$. When $k=n-k$, we have the entry at (0, 0) equal to

$$\alpha_{m_1}\beta_{m_1}\alpha_{m_2}\beta_{m_2}\cdots\alpha_{m_k}\beta_{m_k}$$

Therefore, we have $a_{00}^{(n)}=0$ if n is odd and

$$a_{00}^{(n)}=\sum_{\pi\in\Pi}\alpha_{m_1}\beta_{m_1}\alpha_{m_2}\beta_{m_2}\cdots\alpha_{m_k}\beta_{m_k},$$

where $n=2k$. ▮

Consequently, we have the formula for ψ:

$$\psi^*=\sum_{k=1}^{\infty}\frac{1}{2k}\sum_{\pi}\alpha_{m_1}\beta_{m_1}\alpha_{m_2}\beta_{m_2}\cdots\alpha_{m_k}\beta_{m_k}.$$

Note that $|\alpha_i|<\epsilon$ and $|\beta_i|<\epsilon$ are small. We can now easily obtain approximate formulas of ψ^* and thus, the potential function up to any order we desire. For example, the second order approximate formula of the potential function is

$$\varphi(\bar{x}) \approx -\log|f'(x_0)| - \log \frac{\partial G_0}{\partial x_0}(F(\bar{x})) + \frac{1}{2}(\alpha_0\beta_0 + \alpha_1\beta_1)$$

$$= -\log|f'(x_0)| - \log \frac{\partial G_0}{\partial x_0}(F(\bar{x})) + \frac{1}{2}\left(\frac{\frac{\partial G_0}{\partial x_{-1}}\ \frac{\partial G_{-1}}{\partial x_0}}{\frac{\partial G_0}{\partial x_0}\ \frac{\partial G_{-1}}{\partial x_{-1}}} + \frac{\frac{\partial G_1}{\partial x_0}\ \frac{\partial G_0}{\partial x_1}}{\frac{\partial G_1}{\partial x_1}\ \frac{\partial G_0}{\partial x_0}}\right).$$

6.2.2. The Lattice $\mathbb{Z}^d$ Case

The previous calculation can be extended to the d dimensional lattice case. We assume nearest neighbor interactions:

$$G = (G_i(\bar{x})), \quad i = (i_1, i_2, \ldots, i_d) \in \mathbb{Z}^d,$$

where the value of each component $G_i(\bar{x})$ depends only on variables x_j with

$$|j - i| = |j_1 - i_1| + |j_2 - i_2| + \cdots + |j_d - i_d| \leqslant 1.$$

Since the lattice is of dimension d, the matrix A now is decomposed into a sum of $2d$ infinite matrices:

$$A = L_1 + R_1 + L_1 + R_1 + \cdots + L_d + R_d.$$

For each infinite matrix $L_k = (l_{ij}^{(k)})$, $1 \leqslant k \leqslant d$, its entries are given as follows. Let $i = (i_1, i_2, \ldots, i_{k-1}, i_k, i_{k+1} \cdots, i_d) \in \mathbb{Z}^d$.

$$l_{ij}^{(k)} = \begin{cases} 0 & \text{if } \ j \neq (i_1, i_2, \ldots, i_{k-1}, i_k - 1, i_{k+1}, \ldots, i_d) \\ \dfrac{\partial G_i}{\partial x_j} \Big/ \dfrac{\partial G_i}{\partial x_i} & \text{if } \ j = (i_1, i_2, \ldots, i_{k-1}, i_k - 1, i_{k+1}, \ldots, i_d). \end{cases}$$

Similarly, the entries of the matrix $R_k = (r_{ij}^{(k)})$ are given by

$$r_{ij}^{(k)} = \begin{cases} 0 & \text{if } \ j \neq (i_1, i_2, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_d) \\ \dfrac{\partial G_i}{\partial x_j} \Big/ \dfrac{\partial G_i}{\partial x_i} & \text{if } \ j = (i_1, i_2, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_d). \end{cases}$$

To determine the entry $a_{00}^{(n)}$ of the product

$$A^n = (L_1 + R_1 + L_1 + R_1 + \cdots + L_d + R_d)^n,$$

we expand the right hand side:

$$A^n = \sum_{\pi} C_1 C_2 \cdots C_n,$$

where each C_l, $1 \leqslant l \leqslant n$ is either L_k or R_k and the sum is taken over the direct product of $2d$ symbols over n places.

Note that for either type of matrices L_k or R_k, there is only one non-zero entry on each row or column. Thus, the $(0, 0)$ entry of the product $C_1 C_2 \cdots C_n$ is not zero only if we have the same number of L_k and R_k among $C_1, C_2, \ldots,$ and C_n for every k, $1 \leqslant k \leqslant d$. In particular, the $(0, 0)$ entry is zero if n is odd.

Denote

$$\alpha_i^{(k)} = l_{ij}^{(k)}, j = (i_1, i_2, \ldots, i_{k-1}, i_k - 1, i_{k+1} \cdots, i_d),$$

and

$$\beta_i^{(k)} = r_{ji}^{(k)}, j = (i_1, i_2, \ldots, i_{k-1}, i_k - 1, i_{k+1} \cdots, i_d).$$

Let $\Pi_d = \{\pi = (C_1 C_2 \cdots C_{2m}) : C_l \in \{L_1, \ldots, L_d, R_1, \ldots R_d\}, 0 \leqslant l \leqslant 2m$ and the numbers of L_k's and R_k's are equal for each $k, 0 \leqslant k \leqslant d\}$. We have the following formula.

Proposition 3

$$a_{00}^{(n)} = \begin{cases} 0 & \text{if} \quad n \text{ is odd,} \\ \sum_{\pi \in \Pi_d} \alpha_{i_1}^{(k_1)} \cdots \alpha_{i_m}^{(k_m)} \beta_{i_{m+1}}^{(k_1)} \cdots \beta_{i_{2m}}^{(k_m)} & \text{if} \quad n = 2m, \end{cases}$$

where the sequence of integers $(i_1, i_2, \ldots, i_{2m})$ and $(k_1, \ldots, k_m)$, $1 \leqslant k_l \leqslant d$ are determined by each π.

For example, if $n = 3$ and $\pi = (L_1, L_2, R_1, L_1, R_2, R_1)$, the corresponding term in the sum (the entry at $(0, 0)$ of the matrix $L_1 L_2 R_1 L_1 R_2 R_1$) is

$$l_{0\eta_1}^{(1)} l_{\eta_1\eta_2}^{(2)} r_{\eta_2\eta_3}^{(1)} l_{\eta_3\eta_4}^{(1)} r_{\eta_4\eta_5}^{(2)} r_{\eta_5 0}^{(1)} = \alpha_0^{(1)} \alpha_{\eta_1}^{(2)} \beta_{\eta_3}^{(1)} \alpha_{\eta_3}^{(1)} \beta_{\eta_5}^{(2)} \beta_0^{(1)} = \alpha_0^{(1)} \alpha_{\eta_1}^{(2)} \alpha_{\eta_3}^{(1)} \beta_{\eta_3}^{(1)} \beta_{\eta_5}^{(2)} \beta_0^{(1)},$$

where $0 = (0, \ldots, 0) \in \mathbb{Z}^d$, $\eta_1 = (-1, 0, \ldots, 0)$, $\eta_2 = (-1, -1, 0, \ldots, 0)$, $\eta_3 = (0, -1, 0, \ldots, 0)$, $\eta_4 = \eta_2 = (-1, -1, 0, \ldots, 0)$, and $\eta_5 = \eta_1 = (-1, 0, \ldots, 0)$.

Therefore, we have

$$\psi^* = \sum_{n=1}^{\infty} \frac{1}{2n} \sum_{\pi \in \Pi_d} \alpha_{\pi_1}^{(k_1)} \cdots \alpha_{\pi_n}^{(k_n)} \beta_{\pi_{n+1}}^{(k_1)} \cdots \beta_{\pi_{2n}}^{(k_n)}.$$

The second order approximation of ψ^* can then be easily determined:

$$\psi^* \approx \frac{1}{2}\left(\sum_{k=1}^{d} \alpha_0^{(k)} \beta_0^{(k)} + \alpha_j^{(k)} \beta_j^{(k)} \right),$$

where

$$\alpha_0^{(k)} = l_{0i}^{(k)} = \frac{\partial G_0}{\partial x_i} \Big/ \frac{\partial G_0}{\partial x_0}, \qquad i = (0,..., 0, -1, 0,..., 0) \in \mathbb{Z}^d.$$

$$\beta_0^{(k)} = r_{i0}^{(k)} = \frac{\partial G_i}{\partial x_0} \Big/ \frac{\partial G_i}{\partial x_i}, \qquad i = (0,..., 0, -1, 0,..., 0) \in \mathbb{Z}^d.$$

$$\alpha_j^{(k)} = l_{j0}^{(k)} = \frac{\partial G_j}{\partial x_0} \Big/ \frac{\partial G_j}{\partial x_j}, \qquad j = (0,..., 0, 1, 0,..., 0) \in \mathbb{Z}^d.$$

$$\beta_j^{(k)} = r_{0j}^{(k)} = \frac{\partial G_0}{\partial x_j} \Big/ \frac{\partial G_0}{\partial x_0}, \qquad j = (0,..., 0, 1, 0,..., 0) \in \mathbb{Z}^d.$$

6.3. Spatiotemporal Entropy of Coupled Map Lattices

The main application of the formula of the potential function is to obtain spatiotemporal entropy of coupled map lattices. It is shown that the entropy formula holds when the local hyperbolic set is an attractor.[17] One can directly extend the formula to coupled expanding map lattices. Let $h_\mu(\tau)$ denote the measure theoretical entropy of the $\mathbb{Z}_+^{d+1}$-action τ induced by the map Φ and the spatial translations σ_s with respect to the SRB measure μ. We have

$$h_\mu(\tau) = -\int_{\mathcal{M}} \varphi \, d\mu.$$

The second order approximation is given by

$$h_\mu(\tau) \approx \int_{\mathcal{M}} \left[\log |f'(x_0)| + \log \left| \frac{\partial G_0}{\partial x_0} \right| (F(\bar{x})) - \frac{1}{2} \sum_{k=1}^{d} (\alpha_0^{(k)} \beta_0^{(k)} + \alpha_j^{(k)} \beta_j^{(k)}) \right] d\mu,$$

where $\alpha_0^{(k)}$, $\alpha_j^{(k)}$, $\beta_0^{(k)}$, $\beta_j^{(k)}$ are given in the previous section.

We call this entropy $h_\mu(\tau)$ spatiotemporal entropy since it is the entropy of the $\mathbb{Z}_+^{d+1}$-action τ.

Note that an explicit calculation of $h_\mu(\tau)$ in terms of maps G and F involves an calculation of the SRB measure μ which is still difficult to do. One simple situation is when the local map f is a linear expanding map. The potential function

$$\varphi = -\log |f'| - \log \left|\frac{\partial G_0}{\partial x_0}\right| + \psi^* = -\log p - \log \left|\frac{\partial G_0}{\partial x_0}\right| + \psi^*,$$

where p is the degree of the map f. Therefore,

$$\begin{aligned} h_\mu(\tau) &= \log p + \int \log \left|\frac{\partial G_0}{\partial x_0}\right| d\mu - \int \psi^* \, d\mu \\ &\approx \log p + \int \log \left|\frac{\partial G_0}{\partial x_0}\right| d\mu - \int \frac{1}{2} (\alpha_0 \beta_0 + \alpha_1 \beta_1) \, d\mu. \end{aligned}$$

Thus, in this simple case we have the following conclusion.

The first order of the perturbation of the entropy is due to the local perturbation $\frac{\partial G_0}{\partial x_0}$. *The contribution from the nearest neighbor coupling is at most of the second order in terms of the magnitude of the coupling.*

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